

Entropy and the value of information for investors*

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Abstract: Consider any investor who fears ruin when facing any set of investments that satisfy no-arbitrage. Before investing, he can purchase information about the state of nature in the form of an information structure. Given his prior, information structure α investment dominates information structure β if, whenever he is willing to buy β at some price, he is also willing to buy α at that price. We show that this informativeness ordering is complete and is represented by the decrease in entropy of his beliefs, regardless of his preferences, initial wealth, or investment problem. We also show that no prior-independent informativeness ordering based on similar premises exists.

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1 Introduction

Consider a decision maker operating under uncertainty. When can one say that a new piece of information is more valuable to this agent than another? This question is in general hard to answer, because the ranking of information value typically depends upon at least three considerations. (i) The agent's priors matter. (An agent who is almost convinced that a serious crisis in the strength of the dollar is forthcoming may rank the appearance of bad financial news in China or in Europe very differently than an agent who is less convinced about such a crisis.) (ii) The preferences and wealth of the agent matter. (For the same prior, two agents with different degrees of risk aversion may rank in distinct ways a piece of news that almost eliminates uncertainty about an unlikely financial loss versus a less precise piece of news about more likely events.) (iii) The decision problem to which information is applied matters. (The value of being informed about the likelihood of different risks depends on the availability of insurance markets for these risks.)

The agent possesses some initial prior on a set of payoff-relevant states of nature. An information structure specifies, for every state of nature, a probability distribution over the agent's set of signals, so that each signal leads to an update of the agent's beliefs on the state of nature. The question we address is when one information structure provides more information than another.

The first answer to this fundamental question is provided in the seminal work of Blackwell (1953). According to Blackwell's ordering, an information structure is more informative than another whenever the latter is a garbling of the former, i.e., the less informative signal can be interpreted as observing the more informative signal with noise.¹ Blackwell's Theorem states that this is the case if and only if a decision maker with *any* utility function would prefer to use the former information structure over the latter when facing *any* decision problem. This important result provides a decision-theoretic foundation for Blackwell's informativeness ordering. Of course, the requirement that any decision maker would prefer one information structure to another is very strong, and most pairs of information structures cannot be compared according to Blackwell's ordering. In other words, Blackwell's ordering is very incomplete.

Following recent developments in the theory of riskiness, we attempt here to formulate an approach based on decision-theoretic principles in order to complete Blackwell's ordering.² Restricting our attention to a class of no-arbitrage investment decisions first studied in Arrow (1971a) and to a specific class of ruin-averse utility functions that like to avoid bankruptcy, we postu-

¹Or more precisely, an information structure is a garbling of another one when there exists a stochastic matrix such that the matrix of conditional probabilities of each signal for the less informative structure is obtained by multiplying the matrix for the more informative structure by the stochastic matrix.

²In particular, we follow closely a recent paper by Hart (2011), in which two orderings are proposed to justify the Aumann and Serrano (2008) index of riskiness and the Foster and Hart (2009) measure of riskiness.

late the following informativeness ordering. Fixing a prior over the states, we say that one information structure investment dominates another if, over the allowed class of problems and preferences, whenever the first one is rejected at some price, so is the second. This seems a plausible minimum desideratum for a notion of informativeness.

Our main result is that this informativeness ordering is complete and is represented by the decrease in entropy of the agent's beliefs.³ Specifically, if one considers the prior and the collection of posteriors generated by the information structure, we show that the informativeness of the information structure is represented by the difference between the entropy of the prior distribution and the expected entropy of the conditional posterior distributions.⁴ More precisely, an information structure investment dominates another one in our sense if and only if the entropy decrease resulting from the first is larger than that resulting from the second. Our ordering is complete, since informativeness is characterized by a real number. It is also compatible with Blackwell's ordering, since more information according to Blackwell's ordering necessarily corresponds to decrease in entropy.

An agent's preference over information structures depends highly on the agent's faced risk and on the set of available decisions. It is thus generally impossible to unanimously rank all information structures. Our approach

³Unlike the riskiness papers mentioned in the previous footnote, our decision-theoretic considerations here do not uncover a new index, but provide a new support to the classic concept of entropy.

⁴This difference is referred to as "rate of transmission" by Arrow (1971a).

aims at identifying a wide class of utility functions and investment problems where some version of “unanimity” obtains. In other words, we order information structures by the willingness to pay of agents, who potentially differ in their initial wealth, in their ruin-averse utility functions and in their no-arbitrage investment opportunities. The characterization of this ordering indicates that entropy is the only objective way to speak of the informativeness of information structures: by construction, our ordering is independent both of the agent’s preferences and of the set of available choices, within the classes considered. On the other hand, it is prior-dependent, which may seem to be a limitation. We show, however, that this is unavoidable, since any ordering based on the same postulate as ours is necessarily prior-dependent.⁵ Therefore, in regard to the difficulties described in the first paragraph, we provide an environment in which our complete informativeness ordering takes care of considerations (ii) and (iii), although it cannot possibly do the same with (i).

Section 2 introduces the investment problems that we study, the basic assumptions about the investor, and the notions of an information structure and of valuable information. Section 3 introduces ruin-averse utility functions, no-arbitrage investment sets, and our informativeness ordering, and then proves the main result. Section 4 offers several points of discussion concerning the dependence of the ordering with respect to the agent’s

⁵There are of course other limitations, e.g., our postulate of an objective prior.

prior wealth, presents an axiomatization of our assumptions on preferences and investment sets, and analyzes several examples. Section 5 is a review of the literature, and Section 6 concludes. The Appendix provides most of the proofs, leaving the rest to the online appendix or to our discussion paper version.

2 Investments and Uncertainty

We measure the value of information according to its relevance to investment choices. To this end, we rely on a standard model of investment under uncertainty *à la* Arrow (1971a).⁶

2.1 The Investor

We consider an agent with initial wealth w , and with an increasing and twice differentiable monetary utility function $u: \mathbb{R}_+ \rightarrow \mathbb{R}$. The coefficient of relative risk aversion at wealth $z > 0$ is:

$$\rho(z) = -\frac{u''(z)z}{u'(z)}.$$

We assume that the agent has weakly increasing relative risk aversion (IRRA), namely, that ρ is nondecreasing on \mathbb{R}_+ . This standard assumption is defended from the theoretical point of view by Arrow (1971b). It is also consistent with observed behavior both in the field (Binswanger, 1981; Post, van den Assem,

⁶This basic asset-investment model is used often, e.g., in Mas-Colell, Whinston, and Green (1995), Section 19.E, where it is integrated into a general-equilibrium economy.

Baltussen, and Thaler, 2008) and in laboratory experiments (Holt and Laury, 2002). The class of IRRA utility functions includes the widely used constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA) classes.

We denote by \mathcal{U}_0 the set of such monetary utility functions. For $u \in \mathcal{U}_0$, we let $u(0) = \lim_{z \rightarrow 0} u(z) \in \mathbb{R} \cup \{-\infty\}$.

Let K be the finite set of states of nature. The investor has a prior belief p with full support, fixed throughout the paper.⁷ An investment opportunity or asset is $b \in \mathbb{R}^K$, with the interpretation that if b is taken by an agent with initial wealth w , this wealth becomes $w + b_k$ in state k once uncertainty is realized. We do not allow for bankruptcy (the possibility of negative wealth), and say that an asset b is *feasible* at wealth w when $w + b_k \geq 0$ in every state $k \in K$.⁸

2.2 The Investments

The investor has the opportunity to choose from an investment set B of assets, from which he must take one. One possibility among the available choices is to opt out, namely, to keep his wealth w in a safe asset. We formulate this assumption as $0_K \in B$, where 0_K is the null vector of \mathbb{R}^K . Thus, an investment set consists of a subset of \mathbb{R}^K containing 0_K . For instance, the

⁷Except for Subsection 4.1, where we discuss the impossibility of a prior-independent ordering.

⁸Notice that this is an ex-ante notion of feasibility, which does not take into account the payment of any amount to purchase information. The dominance relation defined later accounts for this.

set B can consist of a set of Arrow securities, or of the asset structure of a complete or incomplete market. Elements in B can be either divisible (for every $b \in B$, $\lambda \in [0, 1]$, $\lambda b \in B$) or indivisible. We say that an investment set B is feasible at w when all its elements are feasible at w .

2.3 Information Structures

An information structure α is given by a finite set of signals S_α , together with probabilities $\alpha_k \in \Delta(S_\alpha)$ for every k . When the state of nature is k , $\alpha_k(s)$ is the probability that the signal observed by the agent is s . It is standard practice to represent any such information structure by a stochastic matrix, with as many rows as states and as many columns as signals; in the matrix, row k is the probability distribution $(\alpha_k(s))_{s \in S_\alpha}$. We assume that every signal s has positive probability under at least one state k . This is without loss of generality, since zero probability signals can be deleted from the set S_α .

It is useful to think of α in terms of a distribution over posterior probabilities. Signal s has a total probability $p_\alpha(s) = \sum_k p(k)\alpha_k(s)$, and the agent's posterior probability on K given s is q_α^s , derived using Bayes' formula:

$$q_\alpha^s(k) = \frac{p(k)\alpha_k(s)}{p_\alpha(s)}.$$

Information structures are ranked according to the Blackwell (1953) ordering, which is a partial order. For this order, a most informative information structure, denoted as $\bar{\alpha}$, is one that perfectly reveals the state of nature k ; hence, for any s , there exists a unique k such that $\bar{\alpha}_k(s) > 0$. A least

informative information structure is any $\underline{\alpha}$ with no informational content: $(\underline{\alpha}_k(s))_{s \in S_{\underline{\alpha}}}$ is the same distribution for all k .

2.4 Valuable Information

Given a utility function u , initial wealth w , a feasible investment set B , and a belief $q \in \Delta(K)$, the maximal expected utility that can be reached by choosing an investment opportunity $b \in B$ is:

$$v(u, w, B, q) = \sup_{b \in B, b \text{ feasible}} \sum_k q(k)u(w + b_k)$$

with the convention that $0 \cdot (-\infty) = 0$. The expected payoff before receiving signal s from α is:

$$\pi(\alpha, u, w, B) = \sum_s p_\alpha(s)v(u, w, B, q_\alpha^s).$$

The possibility of opting out ensures that both $v(u, w, B, q)$ and $\pi(\alpha, u, w, B)$ are always larger than or equal to $u(w)$. The gain $V(\alpha, u, w, B)$ from investment opportunities in B and information α is the difference:

$$V(\alpha, u, w, B) = \pi(\alpha, u, w, B) - u(w).$$

It is often useful for our purposes to rewrite the above expression in this way:

$$V(\alpha, u, w, B) = \sum_s p_\alpha(s)(v(u, w, B, q_\alpha^s) - u(w)).$$

This last expression shows that $V(\alpha, u, w, B) > 0$ if and only if there exists an s such that $v(u, w, B, q_\alpha^s) > u(w)$.

3 Entropy as an Ordering of Information for Investment Problems

3.1 Ruin-Averse Utility and No-Arbitrage Investments

We make two assumptions in our basic framework. One concerns the agent's utility function u and the other, the set B of available assets. These two assumptions, taken together, ensure that the class of utility functions and investment sets are suitable to rank informativeness.

We say that an asset $b \in \mathbb{R}^k$ is *not subject to arbitrage* or is a *no-arbitrage* asset (given initial belief p) if $\sum_k p(k)b_k \leq 0$, and we let B^* be the set of all no-arbitrage assets. An investment set B satisfies no-arbitrage if it contains only no-arbitrage assets ($B \subseteq B^*$), and we let \mathcal{B}^* be the class of no-arbitrage investment sets. The assets not subject to arbitrage are characterized by the property that no risk-averse or risk-neutral agent with belief p would prefer to select such an asset over opting out.⁹ For instance, for an investor with a uniform prior over three states, the asset with payoffs $(-7, 2, 3)$ offers no arbitrage opportunities ex-ante, but it may be a reasonable investment after acquiring appropriate information, e.g., that the true state of nature is not state 1.

We call a monetary utility u *ruin-averse* whenever $u(0) = -\infty$. Thus, a ruin-averse agent is one who prefers to opt out rather than making any

⁹The name *no-arbitrage* assets is appropriate, as they are also characterized by the absence of arbitrage opportunities (see, e.g., Duffie, 1996, Theorem 1, page 4 and the later discussion in section 1.B. of risk-neutral probabilities).

investment that leads to ruin with positive probability. Let $\mathcal{U}^* \subset \mathcal{U}_0$ be the set of all ruin-averse utility functions in our domain. The next lemma (proof in the Appendix), following the analysis of Hart (2011), characterizes ruin aversion by means of coefficients of risk aversion.

Lemma 1 *Let $u \in \mathcal{U}_0$. Then, $u \in \mathcal{U}^*$ if and only if for every $z > 0$ $\rho(z) \geq 1$.*

We discuss the ruin-aversion and no-arbitrage assumptions further in section 4.3

3.2 Information Purchasing

In order to understand the value of information for the agent, we consider a situation in which the agent has the possibility of purchasing information structure α before making an investment decision in B . Decisions whether to purchase information or not are based on the comparison between the expected payoff under the new information and the sure payoff $u(w)$. The agent with utility function u and wealth w purchases information α at price $\mu < w$, given an investment set B , when:¹⁰

$$\pi(\alpha, u, w - \mu, B) \geq u(w).$$

Otherwise, the agent rejects information α at price μ .

Our information ordering is defined as follows:

¹⁰One consequence of assuming that assets are not subject to arbitrage is that no investment is beneficial if no information is acquired, hence the agent's utility is measured by $u(w)$ in this case. See also the NINI property in Subsection 4.3.

Definition 1 *Information structure α investment-dominates (or investment-dominates, for short) information structure β whenever, for every wealth w and price $\mu < w$ such that α is rejected by all agents with utility $u \in \mathcal{U}^*$ at wealth w for every opportunity set $B \in \mathcal{B}^*$, β is also rejected by all those agents.*

The definition can be rewritten: for every w and $\mu < w$,

$$\begin{aligned} & \forall u \in \mathcal{U}^*, B \in \mathcal{B}^*, \pi(\alpha, u, w - \mu, B) < u(w) \\ \implies & \forall u \in \mathcal{U}^*, B \in \mathcal{B}^*, \pi(\beta, u, w - \mu, B) < u(w) \end{aligned}$$

This seems plausible as a minimum desideratum in order to speak of informativeness. The next lemma identifies the important role played in the definition by an agent with a logarithmic utility function:

Lemma 2 *Given an information structure α , a price μ , and a wealth level $w > \mu$, α is rejected by all agents with utility $u \in \mathcal{U}^*$ at wealth level w and price μ given every opportunity set $B \in \mathcal{B}^*$ if and only if α is rejected by an agent with \ln utility at wealth w and price μ for the opportunity set B^* .*

3.3 Entropy Ordering

With the assumptions made about assets and utility functions, our next step is to arrive at a representation of the ordering just defined. In effect, achieving this representation will provide an index of informativeness for information structures, i.e., an objective way to talk about an information

structure being more informative than another, based on the investment framework described. The result below characterizes entropy as such an index, which is independent of the specific utility function of the decision maker, of his wealth, and of the specific investment decision considered. In contrast, such an index cannot be independent of the decision maker's prior, as we show in the next section.

Following Shannon (1948), the entropy of a probability distribution $q \in \Delta(K)$ is the quantity:

$$H(q) = - \sum_{k \in K} q(k) \ln q(k),$$

where $0 \ln(0) = 0$ by continuity.¹¹ The entropy of p is a measure of the level of uncertainty about the state of nature held by the investor with belief p . It is always nonnegative, and is equal to zero only in the case of certainty, i.e., when q puts weight 1 on some state k . It is concave, representing the fact that distributions that are closer to the extreme points in $\Delta(K)$ correspond to a lower level of uncertainty. On the other hand, entropy achieves its global maximum at the uniform distribution, a situation of “maximal uncertainty.”

Recall that following information structure α , (i) the agent's signal is s with probability $p_\alpha(s)$; and (ii) the posterior probability on K following s is q_α^s . The *entropy informativeness* of information structure α is the expected

¹¹For us, any log function would work, including, for example, $\log_2(\cdot)$, which is common in information theory, stemming from the normalization that the amount of information carried by the observation of a Bernoulli random variable with parameter $1/2$ is exactly one bit.

reduction of entropy of the investor's beliefs due to his observation of s . It is this quantity:

$$I(\alpha) = H(p) - \sum_s p_\alpha(s)H(q_\alpha^s).$$

As shown in Subsection 4.1, $I(\alpha)$ depends on p as well as on α . (For notational simplicity, we do not include p as an argument of I . Only in Subsection 4.1 do we make this dependence explicit.) The informativeness is minimal when α is $\underline{\alpha}$ with no informational content, and $I(\underline{\alpha}) = 0$. It is maximal when α is $\bar{\alpha}$ that fully reveals the state of nature k , and value $I(\bar{\alpha}) = H(p)$. Note that given a prior p , I is a numeric index, which hence defines a complete ordering of information structures.

3.4 Main Result

Our main result establishes that the ordering of information structures given by investment dominance coincides with the ordering according to entropy informativeness. Hence this ordering is complete.

Theorem 1 *Information structure α investment-dominates information structure β if and only if $I(\alpha) \geq I(\beta)$.*

The proof of this result hinges on two crucial steps. First, we establish that an agent with logarithmic utility values information about our class of investments using entropy. Then, since by Lemma 2 all agents in the class \mathcal{U}^* reject an information structure α at a price if and only if so does the loga-

rithmic utility, entropy must order information structures in our investment-dominance sense for all agents in \mathcal{U}^* . Notice that although the proof in Lemma 2 borrows from the techniques in Hart (2011), we establish an important additional step, namely, that the logarithmic agent, the universally less risk-averse agent in our class, is also the one who is the most willing to pay for information.

4 Discussion

This section discusses each of the assumptions used in our approach.

4.1 Prior-Independent Ordering

For a given prior p , the informativeness ordering we have introduced is represented by a decrease in entropy. Making the dependence of $I(\alpha)$ on p explicit, we denote here $I(\alpha)$ by $I(\alpha, p)$. An information structure α investment dominates another β , $I(\alpha, p) \geq I(\beta, p)$, if and only if α causes a larger reduction in entropy (from the entropy of the prior p to the expected entropy of the generated posteriors) than does β .

We can prove that there exists no index that orders information structures that is both compatible with investment dominance and independent of the agent's prior. In order to do this, let us define the following.

Definition 2 *An information structure α investment-dominates independently of the prior β , whenever α investment-dominates β for all priors p .*

This definition turns out to be too strong a requirement, and leads to the following impossibility result:

Theorem 2 *There exists no numerical representation that orders information structures according to the ordering of investment dominance independently of the prior.*

The proof of this theorem can be found in Cabrales, Gossner, and Serrano (2011) or in the online Appendix.

4.2 Uniformity in Wealth

After the discussion opened in the previous subsection, we now return to fixing a prior p , which will remain fixed for the rest of the paper.

We have defined information dominance as wealth-independent, but this is not really a restriction. To see this, consider the following alternative definition:

Definition 3 *Information structure α investment-dominates information structure β for wealth w if, for every price $\mu < w$ such that α is rejected by all agents with utility $u \in \mathcal{U}^*$ for every opportunity set $B \in \mathcal{B}^*$, β is also rejected by all those agents.*

In mathematical terms: for every $\mu < w$,

$$\begin{aligned} & \forall u \in \mathcal{U}^*, B \in \mathcal{B}^*, \pi(\alpha, u, w - \mu, B) < u(w) \\ \implies & \forall u \in \mathcal{U}^*, B \in \mathcal{B}^*, \pi(\beta, u, w - \mu, B) < u(w) \end{aligned}$$

This definition leads to the following theorem:

Theorem 3 *Information structure α investment-dominates information structure β for wealth w if and only if $I(\alpha) \geq I(\beta)$.*

The result clearly follows because Lemma 2 holds for either Definition 2 or Definition 3 and the ordering $I(\cdot)$ induced by logarithmic preferences is independent of wealth.

In a similar vein, we have made so far comparisons for agents with the same level of wealth. This can be avoided provided the pricing is done as a proportion of wealth, as we show in Cabrales, Gossner, and Serrano (2011).

4.3 On the Ruin-Aversion and No-Arbitrage Assumptions

This subsection is intended to shed additional light on the role of the ruin-averse utility functions – the class \mathcal{U}^* – and the class of investment sets not subject to arbitrage – \mathcal{B}^* . So far, the assumptions underlying our main result, Theorem 1, were that (1) the decision makers we study strongly dislike situations in which their wealth approaches zero (ruin aversion), and (2) the investments they consider do not offer profitable opportunities in the absence of new additional information (no arbitrage). Now we offer a joint axiomatization of such economic circumstances, i.e., of such preference-investment pairs.

Thus, consider in general any class of utility functions included in our original IRRA class \mathcal{U}_0 and any class \mathcal{B} of feasible investment sets. First, it is worth pointing out that the classes \mathcal{U}^* of ruin-averse functions and \mathcal{B}^* of no-arbitrage sets jointly have two properties that we now turn to discuss.

The first property is *No Investment under No Information*, or NINI for short. According to this property, in the absence of information beyond the prior p , the agent prefers to opt out rather than to invest in risky elements of B . As stated, it is a joint assumption on the possible investment set B and the agent's utility function u . It can be viewed as a normalization: for a decision maker who is considering improving his information before investing, we define his initial position as “not being ready to invest” if he gets no new information.

The NINI property expresses the idea that B is such that the gain from investment opportunities is null unless α has some informational content; more precisely:

NINI: \mathcal{B} is the class of investment sets B such that for

$$V(\underline{\alpha}, u, w, B) = 0 \text{ for every } u \in \mathcal{U}_0, w \in \mathbb{R}_+.$$

To motivate the second property, we now discuss the circumstances under which information is valuable to the agent. First, note that if B does not contain feasible elements b such that $b_k > 0$ in some state k , the agent always weakly prefers to opt out. More generally, an agent who fully learns that k

is the state of nature cannot take advantage of such information unless there exists a feasible b offering a gain in state k . We say that an investment set B is *investment-prone* if, for every $k \in K$, there exists $b \in B$ such that $b_k > 0$.

What quality of information is needed to ensure that every investor takes advantage of investment-prone sets? We say that information structure α is *sometimes certain* if $q_\alpha^s(k) = 1$ for some k and s , that is, when there is a signal s which, if received, reveals that the state of nature is k for sure. If α is not sometimes certain, we call it *always uncertain*.

The next lemma shows that *sometimes-certain* information structures are always advantageous, provided B is feasible and investment-prone.

Lemma 3 *If B is investment-prone and feasible at wealth level w , then $V(u, w, \alpha, B) > 0$ for every α that is sometimes certain and for every $u \in U_0$.*

The second joint property of utility functions and investment sets is that only investors with access to *sometimes-certain* information structures are *always inclined to invest*. This property, written as SCAI for short, is expressed as follows:

SCAI: \mathcal{U} consists of the elements u of \mathcal{U}_0 satisfying the condition that there exists a wealth level w and an investment-prone set B of feasible investment opportunities such that $V(\alpha, u, w, B) = 0$ for every always-uncertain α .

According to SCAI, whenever α is *always uncertain*, then there exists a feasible and investment-prone set of investment opportunities such that the agent weakly prefers to opt out. The idea is that not every piece of information is always valuable, i.e., valuable for every agent in every circumstance. In particular, risk-averse agents, like ours, may not use investment opportunities with a positive expected profit if the associated risk is too high. On the other hand, our agents surely can take advantage of being fully informed. The SCAI property establishes a restriction on the class of agents, requiring that only sometimes-certain information structures be *always* valuable to them.

As noted above, both the classes \mathcal{U}^* of ruin-averse utility functions and \mathcal{B}^* of no-arbitrage investment sets satisfy NINI and SCAI. That they satisfy NINI is clear: no-arbitrage assets offer no profitable investment opportunity to risk-averse agents if there is no new information. To see that they also satisfy SCAI, think of a typical investment-prone asset with large negative payoffs in all states but one; with such no-arbitrage assets around, a risk-averse investor will want to opt out unless the new information completely reveals one state. Taken together, NINI and SCAI therefore depict situations in which a risk-averse investor is cautious in utilizing new information, given that the available investments out there may include very risky deals.

What is perhaps more surprising is that these two properties uniquely define a set \mathcal{U} of utility functions and a class \mathcal{B} of investment sets, and that

both $\mathcal{U} = \mathcal{U}^*$ and $\mathcal{B} = \mathcal{B}^*$:

Theorem 4 \mathcal{U} and \mathcal{B} satisfy NINI and SCAI if and only if \mathcal{U} is the class \mathcal{U}^* of ruin-averse utility functions, and \mathcal{B} is the class \mathcal{B}^* of no-arbitrage investment sets.

The proof of the theorem can be found in Cabrales, Gossner, and Serrano (2011).

4.4 An example

We present an example that illustrates how our framework serves to complete Blackwell's ordering.

Example 1 Let $K = \{1, 2, 3\}$ and fix a uniform prior. Consider two information structures that are not ordered in the Blackwell sense. For instance, let each of the two information structures have two signals:

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.9 \\ 0 & 1 \end{bmatrix}$$

To see that they are not ranked according to Blackwell, we exhibit two decision problems where a decision maker would rank them differently. For instance, in Problem 1 the agent must choose one of two actions: action 1 gives a utility of 1 only in the first two states, and 0 otherwise, while action 2 gives a utility of 1 only in the third state, and 0 otherwise. Problem 2, in contrast, has action 1 pay a utility of 1 only in the first state, and 0 otherwise,

while action 2 gives a utility of 1 only in states 2 or 3, and 0 otherwise. Facing Problem 1, the decision maker would value α_1 more than α_2 : following the first signal in α_1 , he would choose the first action and following the second signal in α_1 , he would choose the second action, thereby securing a utility of 1. This would be strictly greater than his utility after α_2 . On the other hand, facing Problem 2, he would under α_2 choose action 1 after the first signal and action 2 after the second, yielding a utility of $29/30$, which is greater than his optimal utility after α_1 .

But by calculating their entropy reduction from the uniform prior, we know that $I(\alpha_1) > I(\alpha_2)$. Thus, for every investment problem we consider and every utility function in our allowable class, the first information structure is more valuable –more informative– than the second when starting from a uniform prior. The difficulty in the two problems of this example is that, if one specifies an economic environment like ours to make sense of the action-utility pairs provided, the resulting investment set fails to satisfy no-arbitrage, for at least some wealth levels.

We refer the reader to Cabrales, Gossner, and Serrano (2011) for several more examples, illustrating optimization over investment sets, how entropy ranks lotteries over information structures, and how the framework extends to a continuum of signals.

5 Related Literature

Previous authors have justified the use of the entropy index based on information theoretic considerations, and have shown that it arises naturally in a variety of dynamic setups. A salient feature of our work is that it shows that entropy is also rooted in economic and decision-theoretic arguments in static setups.

Shannon (1948), who introduced entropy as a measure of information, characterizes it as the only measure that jointly satisfies these three properties: continuity, monotonicity, and decomposability. Marschak (1959) presents formal arguments in favor of using entropy in the study of the demand for information and its cost. After these classic contributions, the concept has arisen separately in several fields of economics, and we provide only a brief partial survey here.

Gossner, Hernández, and Neyman (2006) study repeated games in which one of the players, who can forecast the realization of future states of nature, can transmit information to others through his choice of actions. They provide a closed-form characterization, based on entropy, of the set of distributions that the players can achieve.

Gossner and Tomala (2006) analyze games in which a team of players uses private signals in a repeated game to secretly coordinate their actions. They show that entropy is an adequate measure of informativeness to study

the trade-off between the generation of signals for future coordination and the use of acquired secret information.

The Theil coefficient for economic inequality is based on the entropy of observed data. Bourguignon (1979) axiomatizes this measure by showing that it is the only one that is consistent with a property of income-weighted decomposability.

In Sims (2003)'s model of rational inattention, entropy is used to measure information acquisition by agents with bounded information-processing capabilities. This approach has been applied to different economic problems. For instance, Peng (2005) explores its implications for asset-price dynamics and consumption behavior. Sims (2005) offers a summary of other contributions in this area.

Arrow (1971a) considers an investor who has access to a set of securities that pay a positive amount in only one state of nature. He shows that, if the value of information about the state is independent of the returns, then this value is given by the entropy of this information.

Entropy is also the basis for the relative-entropy measure of proximity of probability distributions. Blume and Easley (1992) and Sandroni (2000) show that in dynamic exchange economies, markets favor agents who make the most accurate predictions when accuracy is measured according to relative entropy. Other applications of relative entropy include ambiguity aversion (Maccheroni, Marinacci, and Rustichini, 2006) and reputation models

(Gossner, 2011).

Measuring the amount of information is a common problem in economics and decision theory.¹² Most of the work in this area follows the seminal work of Blackwell (1953). For Blackwell, an information structure α is more informative than an information structure β if every decision maker prefers α to β in any decision problem. As noted in the introduction, the main drawback of this approach is that this criterion does not provide a complete ordering. Researchers have made progress by focusing on decision makers who have preferences in a particular class. Lehmann (1988), for instance, restricts the analysis to problems that generate monotone decision rules (and hence satisfy single-crossing conditions). A similar approach is followed by Persico (2000), and Levin and Athey (2001). Jewitt (2007) clarifies the different versions of monotonicity used by Lehmann (1988), Persico (2000) and Levin and Athey (2001). We follow this tradition with two main differences. First, unlike the measures in those papers, our measure of informativeness provides a complete order of all information structures. Second, we achieve this through a different kind of restriction on admissible preference orderings, and we characterize decision problems in terms of investment opportunities, thereby restricting the framework.

Gilboa and Lehrer (1991) and Azrieli and Lehrer (2008) take an approach that differs markedly with respect to the one used in papers cited in the

¹²Veldkamp (2011) provides a good summary of ways in which economists have measured informativeness and its applications.

previous paragraphs. Rather than choosing a class of decision problems, and then providing an ordering of information structures, they characterize the orderings that are possible for any prespecified class of decision problems. The 1991 paper considers deterministic information structures, and the 2008 paper extends the analysis to stochastic ones. The entropy function satisfies the axioms of the first paper, and hence it is a “value of information” function over partitions of the set of states. The 2008 paper shows that reducibility, weak order, independence, continuity, and convexity characterize all binary relations on information structures induced by decision problems, entropy being one of them.

A recent paper by Ganuza and Penalva (2010) provides a different way to order information structures (also a partial order) which is based not on decision-theoretic considerations, but rather on various measures of dispersion of distributions. (Many of those measures are presented in Shaked and Shanthikumar, 2007). They show that while some of their measures are implied by notions of informativeness based on the value of information, the strongest of their criteria, supermodular precision, is strictly different: it neither implies nor is implied by those notions of informativeness. They then proceed to study the implications of greater informativeness (in their sense) for auction problems, and show that while greater informativeness improves allocational efficiency, the auction organizers are not always interested in increasing informativeness since that may increase the buyers’ informational

rents.

6 Conclusion

In the classic framework of information structures proposed by Blackwell, we have found that, for a given prior, a natural informativeness ordering (namely, that if a decision maker is willing to pay a price for an information structure, he is willing to pay that price for a one that dominates it) is complete when considered over the class of ruin-averse utility functions and no-arbitrage investment sets. Furthermore, this ordering is represented by the expected decrease of entropy from the prior to the posteriors, and this ordering is complete. We have also found that no such ordering can be made independent of the decision maker's prior.

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A Proofs

A.1 Proof of Lemma 1

We follow Hart (2011). Assume that for every $z > 0$,

$$\rho(z) = -\frac{u''(z)}{u'(z)}z \geq 1.$$

By integrating between $z < 1$ and 1 we obtain,

$$\ln u'(z) - \ln u'(0) \geq -\ln(z),$$

which can be rewritten as:

$$u'(z) \geq \frac{u'(0)}{z}.$$

A second integration between $z < 1$ and 1 shows that

$$u(z) - u(1) \leq u'(0) \ln(z),$$

and hence that $u(0) = \lim_{z \rightarrow 0} u(z) = -\infty$.

Now assume that there exists $z_0 > 0$ where $\rho(z_0) < 1$. Since u is IRRA, then for every $z \leq z_0$, $\rho(z) \leq \rho(z_0) < 1$. Integrating shows that for every $z \leq z_0$,

$$\ln u'(z) - \ln u'(z_0) \leq -\rho(z_0)(\ln(z) - \ln(z_0)),$$

which can be expressed as

$$u'(z) \leq u'(z_0) \left(\frac{z}{z_0} \right)^{-\rho(z_0)}.$$

A second integration between $z < z_0$ and z_0 shows:

$$u(z) - u(z_0) \geq \frac{z_0 u'(z_0)}{1 - \rho(z_0)} \left(\left(\frac{z}{z_0} \right)^{1 - \rho(z_0)} - 1 \right).$$

Since $1 - \rho(z_0) > 0$, the limits of the right-hand side, and hence of the left-hand side, are finite. This shows that $u(0) > -\infty$.

A.2 Proof of Lemma 2

First, note that the *only if* condition is satisfied since the \ln utility function belongs to U^* and $B^* \in \mathcal{B}^*$.

We now prove the *if* part. Assume that α is rejected at price μ given the investment set B^* by an agent with \ln utility and with wealth w . For

$u \in \mathcal{U}^*$, Lemma 1 shows that $\rho(z) \geq 1$ for $z > 0$; hence:

$$\frac{u''(z)}{u'(z)} \leq -\frac{1}{z}.$$

By integration between w and z :

$$\begin{cases} \ln u'(z) - \ln u'(w) \leq -\ln(z) + \ln(w) & \text{if } z \geq w; \\ \ln u'(z) - \ln u'(w) \geq -\ln(z) + \ln(w) & \text{if } z \leq w. \end{cases}$$

Once w is fixed, a second integration with respect to z between w and z' shows that for every z' ,

$$u(z') - u(w) \leq wu'(w)(\ln(z') - \ln(w)).$$

Hence, given any belief q , $B \in \mathcal{B}^*$ and $\mu < w$, we can write:

$$v(u, w - \mu, B, q) - u(w) \leq wu'(w)(v(\ln, w - \mu, B, q) - \ln(w));$$

and by summation over q_α , for every $B \in \mathcal{B}^*$ and $\mu < w$, we obtain:

$$\pi(\alpha, u, w - \mu, B) - u(w) \leq u'(w)w(\pi(\alpha, \ln, w - \mu, B) - \ln(w)).$$

Since $\pi(\alpha, u, w - \mu, B)$ is nondecreasing in B , and B^* is the maximal element of \mathcal{B}^* , then for every $B \in \mathcal{B}^*$ and $\mu < w$ we have:

$$\pi(\alpha, u, w - \mu, B) - u(w) \leq wu'(w)(\pi(\alpha, \ln, w - \mu, B^*) - \ln(w)) < 0,$$

which is the desired conclusion.

A.3 Proof of Theorem 1

Lemma 2 shows that α investment-dominates β if and only if, for every w and $\mu < w$, an agent with \ln utility function who rejects α for the opportunity set B^* also rejects β . The following lemma characterizes the value of information for an agent with \ln utility function and opportunity set B^* .

Lemma A.1 *For every $w > 0$ and belief q ,*

1.

$$v(\ln, w, B^*, q) = \ln(w) - H(q) - \sum_k q(k) p(k).$$

2.

$$\pi(\alpha, \ln, w, B^*) = I(\alpha) + \ln(w).$$

Proof. For the first point, $v(\ln, w, B^*, q)$ is the maximum of $\sum_k q(k) \ln(w + b_k)$ over (b_k) such that $\sum_k p(k) b_k \leq 0$. The first-order condition shows that $w + b_k$ is proportional to $\frac{q_k}{p_k}$, and hence equal to $w \frac{q_k}{p_k}$. We then obtain:

$$\begin{aligned} v(\ln, w, B^*, q) &= \ln(w) + \sum_k q(k) \ln q(k) - \sum_k q(k) \ln p(k) \\ &= \ln(w) - H(q) - \sum_k q(k) \ln p(k). \end{aligned}$$

For the second point, we rely on the previous expression to deduce:

$$\begin{aligned} \pi(\alpha, \ln, w, B^*) &= \sum_s p_\alpha(s) v(\ln, w, B^*, q_\alpha(s)) \\ &= \ln(w) - \sum_s p_\alpha(s) H(q_\alpha^s) - \sum_{k,s} p_\alpha(s) q_\alpha^s(k) \ln p(k) \\ &= I(\alpha) + \ln(w) \end{aligned}$$

since $\sum_s p_\alpha(s) q_\alpha^s(k) = p(k)$. ■

We now complete the proof of Theorem 1. Recall that by Lemma A.1, an agent with utility function \ln rejects α at price $\mu < w$ for the opportunity set B^* if and only if:

$$I(\alpha) < \ln \left(\frac{w}{w - \mu} \right).$$

If $I(\alpha) \geq I(\beta)$, then β is rejected whenever α is. If, on the contrary, $I(\alpha) < I(\beta)$, let μ be such that:

$$I(\alpha) < \ln \left(\frac{w}{w - \mu} \right) < I(\beta).$$

At this price μ , α is rejected whereas β is accepted. Hence, α does not investment-dominate β .

A.4 Proof of Lemma 3

Let k, s be such that $q_\alpha^s(k) = 1$, and let $b \in B$ be such that $b_k > 0$. If b_k is feasible, then we have $v(u, w, B, q_\alpha^s) \geq u(w + b_k) > u(w)$. Hence,

$$V(\alpha, u, w, B) \geq p_\alpha(s)(u(w + b_k) - u(w)) > 0.$$

B Appendix: Further proofs (not for publication)

B.1 Proof of Theorem 2

By Theorem 1 and a fixed prior p , the only possible index is $I(\alpha, p) = H(p) - \sum_s p_\alpha(s)H(q_\alpha^s)$. Therefore, it suffices to construct an example to show that this index orders two information structures in different ways for two different priors. The example follows.

Let $K = \{1, 2, 3\}$. Let $p_1 = (1/2, 1/2, 0)$ and $p_2 = (1/3, 1/3, 1/3)$ and an agent with $u(x) = \ln(x)$.

Let information structures α_1 and α_2 be described by these two-signal three-state matrices:

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & 0.5 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 0.3 & 0.7 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, the expected utility for the agent with logarithmic utility under α_1 is larger than that for α_2 when priors are p_1 as the former gives her full information while the latter does not. It thus follows that:

$$I(\alpha_1, p_1) = H(p_1) - \sum_s p_{\alpha_1}(s)H(q_{\alpha_1}^s) > H(p_1) - \sum_s p_{\alpha_2}(s)H(q_{\alpha_2}^s) = I(\alpha_2, p_1).$$

What is the expected entropy of the posteriors generated by α_1 and α_2 under p_2 ? First, the utility for a ln agent of prior p_2 is $\ln(1/3)$. Then for α_1

the expected utility is:

$$\left(\frac{2}{3}\right) \ln\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right) \ln\left(\frac{1}{3}\right) = -0.63651.$$

Therefore,

$$H(p_2) - \sum_s p_{\alpha_1}(s)H(q_{\alpha_1}^s) = \frac{(2/3) \ln 2}{\ln 2} = \frac{0.46210}{\ln 2} = 2/3.$$

As for α_2 , the (conditional on p_2) probability of either signal is $13/30$ and $17/30$. After she observes each signal, her posteriors are $(3/13, 0, 10/13)$ and $(7/17, 10/17, 0)$, respectively. Thus, her expected \ln utility from α_2 is:

$$\begin{aligned} & \left(\frac{13}{30}\right) \left(\left(\frac{3}{13}\right) \ln\left(\frac{3}{13}\right) + \left(\frac{10}{13}\right) \ln\left(\frac{10}{13}\right) \right) \\ & + \left(\frac{17}{30}\right) \left(\left(\frac{7}{17}\right) \ln\left(\frac{7}{17}\right) + \left(\frac{10}{17}\right) \ln\left(\frac{10}{17}\right) \right) = -0.618. \end{aligned}$$

Noting that $\ln(1/3)$ is the expected utility from the prior, we can derive:

$$H(p_2) - \sum_s p_{\alpha_2}(s)H(q_{\alpha_2}^s) = \frac{0.48061}{\ln 2}.$$

That is,

$$I(\alpha_1, p_2) = H(p_2) - \sum_s p_{\alpha_1}(s)H(q_{\alpha_1}^s) < H(p_2) - \sum_s p_{\alpha_2}(s)H(q_{\alpha_2}^s) = I(\alpha_2, p_2).$$

Hence, whereas for prior p_1 information structure α_1 is more informative than α_2 , the opposite is true for prior p_2 . ■

B.2 Proof of Theorem 4

Given a class $\mathcal{U} \subseteq U_0$ of utility functions, we say that an investment b individually satisfies NINI if for every $u \in \mathcal{U}$ and $w \in \mathbb{R}_+$ such that b is feasible,

$$\sum_k p(k)u(w + b_k) \leq u(w).$$

Thus, b individually satisfies NINI when, under no information, the agent does not prefer b to opting out. We denote as \tilde{B} the set of investments that individually satisfy the NINI property. Since 0_K satisfies NINI, the NINI investment set is a nonempty investment set.

Lemma B.1 *Given \mathcal{U} , \mathcal{B} satisfies NINI if and only if \mathcal{B} is the class of investment sets contained in \tilde{B} .*

Proof. \mathcal{B} satisfies NINI if and only if it contains all the investment sets B such that for every $w > 0$ and $u \in \mathcal{U}$,

$$V(\underline{\alpha}, u, w, B) = 0.$$

That is, if B is such that for every $w > 0$, $u \in \mathcal{U}$, it is true that

$$\sup_{b \in B, b \text{ feasible}} \sum_k p(k)u(w + b_k) = 0.$$

An equivalent way to write the previous statement is: for every $w > 0$, $u \in \mathcal{U}$ and $b \in B$ feasible, then we have:

$$\sum_k p(k)u(w + b_k) \leq u(w),$$

which is finally equivalent to $B \subseteq \tilde{B}$. ■

Therefore, we assume from this point on that \tilde{B} is the NINI investment set corresponding to a set of utility functions \mathcal{U} , and that \mathcal{B} is the class of investment sets contained in \tilde{B} .

We say that a set $A \subseteq \mathbb{R}^K$ is *comprehensive* if, for every feasible b' and for every feasible $b \in A$ such that $b'_k \leq b_k$ for every k , we also have $b' \in A$.

Lemma B.2 \tilde{B} is comprehensive.

Proof. Assume that $b \in \tilde{B}$ and that b' is such that $b'_k \leq b_k$ for every k . Then, b is feasible at wealth w whenever b' is; and for every $u \in \mathcal{U}, w \in \mathbb{R}_+$,

$$\sum_k p(k)u(w + b'_k) \leq \sum_k p(k)u(w + b_k) \leq u(w).$$

Hence, $b' \in \tilde{B}$. ■

We observe that if \tilde{B} is not investment-prone, neither is any element of \mathcal{B} , a subset of \tilde{B} . In this case, SCAI becomes trivially equivalent to $\mathcal{U} = \emptyset$. In contrast, the following proposition characterizes SCAI when \tilde{B} is investment-prone.

Proposition B.3 If \tilde{B} is investment-prone, then \mathcal{U} satisfies SCAI if and only if $\mathcal{U} = \mathcal{U}^*$.

Proof. We divide the proof into a series of lemmata.

Lemma B.4 *Let $u \in \mathcal{U}_0$. If $u(0) > -\infty$, then for every w and for every B that is investment-prone and feasible, there exists an always-uncertain α such that*

$$V(u, w, \alpha, B) > 0.$$

Proof. Fix u, w , and the set B that is investment-prone and feasible.

For $1 > \varepsilon > 0$, let α^ε be defined by $S_{\alpha^\varepsilon} = K$, $\alpha_k^\varepsilon(s) = 1 - \varepsilon$ if $k = s$, and $\alpha_k(s) = \frac{\varepsilon}{K-1}$ otherwise. It can easily be verified that α^ε is *always uncertain* for every $\varepsilon > 0$, and that as $\varepsilon \rightarrow 0$, $q_{\alpha^\varepsilon}^k(k) \rightarrow 1$ for every s .

Since B is investment-prone, there exist k^* and $b^* \in B$ such that $b_{k^*}^* > 0$.

We now have

$$\begin{aligned} v(u, w, B, q_{\alpha^\varepsilon}^{k^*}) &= \sup_{b \in B} \sum_k q_{\alpha^\varepsilon}^{k^*}(k) u(w + b_k) \\ &\geq \sum_k q_{\alpha^\varepsilon}^{k^*}(k) u(w + b_k^*) \\ &\geq q_{\alpha^\varepsilon}^{k^*}(k^*) u(w + b_{k^*}^*) + (1 - q_{\alpha^\varepsilon}^{k^*}(k^*)) u(0). \end{aligned}$$

Hence,

$$\lim_{\varepsilon \rightarrow 0} v(u, w, B, q_{\alpha^\varepsilon}^{k^*}) = u(w + b_{k^*}^*) > u(w),$$

which shows that for ε small enough, $v(u, w, B, q_{\alpha^\varepsilon}^{k^*}) > 0$ and therefore $V(u, w, \alpha^\varepsilon, B) > 0$. ■

Lemma B.5 *Let $u \in \mathcal{U}_0$ and assume that \tilde{B} is investment-prone. If $u(0) = -\infty$, then there exist w and an investment-prone set B that is feasible at w*

such that

$$V(u, w, \alpha, B) = 0 \text{ for every always-uncertain } \alpha.$$

Proof. Since \tilde{B} is investment-prone, for every $k \in K$ there exists b^k such that $b^k > 0$. Let $b^+ = \min_k b^k > 0$, and $b^- = \min(\min_{k \neq j} b^k, -1) < 0$. The investment b^k given by $b_k^k = b^+$ and $b_j^k = b^-$ for every $j \neq k$ is such that for every j , $b_j^k \leq b_j^k$. Since \tilde{B} is comprehensive from Lemma B.2, it follows that $b^k \in \tilde{B}$. Let B be the investment-prone set $B = \{0_K\} \cup \{b^k, k \in K\}$. B is feasible at wealth-level $w = -b^-$. Let α be *always uncertain* and assume $u(0) = -\infty$. For every $s \in S_\alpha$ and for every $b^k \in B$, the expected utility from investing in b^k conditional on s is

$$\begin{aligned} \sum_{k'} q_\alpha^s(k') u(w + b_{k'}^k) &= q_\alpha^s(k) u(w + b^+) + (1 - q_\alpha^s(k)) u(w + b^-) \\ &= q_\alpha^s(k) u(w + b^+) + (1 - q_\alpha^s(k)) u(0) \\ &= -\infty. \end{aligned}$$

Thus, for every $s \in S_\alpha$,

$$v(u, w, B, q_\alpha^s) = u(w),$$

which implies that:

$$V(u, w, \alpha, B) = 0.$$

■

Lemmata B.4 and B.5 provide the proof of Proposition B.3. ■

Lemma B.6 *If $\mathcal{U} = \mathcal{U}^*$, then the only class \tilde{B} that satisfies NINI is the class \mathcal{B}^* of investment sets not subject to arbitrage.*

Proof. We need to show that the NINI investment set \tilde{B} coincides with the set \mathcal{B}^* of assets not subject to arbitrage.

For any $b \in \mathcal{B}^*$, and for any $u \in \mathcal{U}^*$ and w such that b is feasible, u is concave and increasing. This implies:

$$\sum_k p(k)u(w + b_k) \leq u(w + \sum_k p(k)b_k) \leq u(w),$$

and hence, $b \in \tilde{B}$.

Now consider $b \in \tilde{B}$. Note that u given by $u(z) = \ln(z)$ for $z > 0$ is in \mathcal{U}^* . Hence, $b \in \tilde{B}$ implies that for every w large enough,

$$\sum_k p(k) \ln(w + b_k) \leq \ln(w),$$

which is equivalent to

$$\sum_k p(k) \ln\left(1 + \frac{b_k}{w}\right) \leq 0.$$

Hence, for every $\varepsilon > 0$ small enough,

$$\sum_k p(k) \ln(1 + \varepsilon b_k) \leq 0.$$

A first-order Taylor expansion shows that this implies

$$\sum_k p(k)b_k \leq 0,$$

and hence, $b \in B^*$. ■

To wrap up the proof of Theorem 4, assume first that \mathcal{U} and \mathcal{B} satisfy NINI and SCAI. Then from Lemma B.1, \mathcal{B} is the class of investment sets contained in \tilde{B} . If \tilde{B} is not investment-prone, then $\mathcal{U} = \emptyset$, in which case $\tilde{B} = \mathbb{R}^K$, a contradiction. Hence, \tilde{B} is investment-prone, and from Proposition B.3, $\mathcal{U} = \mathcal{U}^*$. Finally, it follows from Lemma B.6 that $\mathcal{B} = \mathcal{B}^*$.

We now show that \mathcal{U}^* and \mathcal{B}^* satisfy NINI and SCAI. With the assumption that $\mathcal{B} = \mathcal{B}^*$, \mathcal{U}^* satisfies SCAI from Proposition B.3. With $\mathcal{U} = \mathcal{U}^*$, \mathcal{B}^* satisfies NINI from Lemma B.6. ■