Communication and Money in Repeated Games on Networks

Alexander Wolitzky

Stanford University and Microsoft Research

June 7, 2012

Abstract

A key obstacle to coordination and cooperation in many networked environments is that behavior in each bilateral relationship is not observable to individuals outside that relationship: that is, information is *local*. This paper investigates when players can use communication to replicate any outcome that would have been sustainable were this information public. A benchmark result is that public information can always be replicated if public communication is possible, but that if only local communication is possible then public information can only be replicated if the network is 2-connected. The main result is that public information can always be replicated with local communication if the players also have access to *fiat money*, which is modeled as undifferentiated tokens that can be freely transferred among linked players. It is shown that such money is *essential* if the network contains a "nice" subnetwork: in leading applications, this condition reduces to the property that the network contains a subtree of size at least three.

Very Preliminary. Comments Welcome.

1 Introduction

Consider three people—1, 2, and 3—arranged on a line: 1 and 2 have a relationship, and 2 and 3 have a relationship, but 1 and 3 do not. In this situation, 1 and 3 might hope to keep 2 on good behavior by threatening "community enforcement": if 2 cheats 1, then 3 cheats 2. But if 2 cheats 1, how does 3 find out? She doesn't have a relationship with 1, and 2 clearly can't be trusted to tell her. So the group has a problem.

Their problem is quite similar to the classical problem of trade absent a double coincidence of wants. Suppose 1 is a university, 2 is an economist, and 3 is a barber. Again, community enforcement looks promising: if the economist doesn't give a lecture, she doesn't get a haircut. But the barber doesn't know if she gave the lecture, and he can't trust her to reveal that information to him.[1]

In both of these examples, the obstacle to sustaining cooperation is that information about individuals' past behavior in a bilateral relationship is *local*: it is common knowledge within the relationship, but is not observable to outsiders. In addition, letting the players communicate locally does not allow them to sustain certain outcomes—like *(lecture,haircut)*—that would have been sustainable if this information were public to all players. In the language of this paper, local communication does not *replicate* public information in these examples.

The goal of this paper is to compare different communication technologies in terms of their ability to replicate public information. A benchmark result is that public information can always be replicated if public cheap talk is possible, but that if only private cheap talk with one's neighbors is possible then public information can be replicated only if the network is 2-connected (i.e., it remains connected after any node is removed).[2] The intuition is simple: With public cheap talk, players can be asked to report their observations, and can be punished if their reports conflict.[3] As long as information is common knowledge within each bilateral relationship—as I assume—an individual player cannot benefit from lying.

[1] This example is adapted from Mishkin (1998), quoted in Wallace (2001).

[2] The former result is reminiscent of the theorem of Ben-Porth and Kahneman (1996), and the latter result is reminiscent of the theorem of Renault and Tomala (1998). The precise connection between these results is discussed later.

[3] I restrict attention to games with a mutual-minmax Nash equilibrium, so designing punishments is trivial.

With private cheap talk, 2-connectedness implies that there are at least two independent paths through which a piece information can reach each player, so again a single player cannot gain from lying.

In general, however, public information cannot be replicated with only private cheap talk when the network is not 2-connected, as the above examples show. I therefore consider the case where players can not only talk privately but can also privately exchange *evidence*. I allow a fairly minimal form of evidence: the players are endowed with undifferentiated, divisible tokens that they can freely transfer to their neighbors. A key feature of tokens is that a player cannot send another more tokens than she has; in contrast, a player can always send any cheap talk message. I view these tokens as a natural model of fiat money, and therefore henceforth refer to them as *money*.[4]

The main result of the paper is that public information can always be replicated with private cheap talk and money. Very roughly speaking, the idea is to endow "leaf players"—like 1 and 3 in the examples—with money, and to endow "non-leaf players"—like 2—with none. Non-leaf players must then obtain money from leaf players in order to convince others that they have behaved well, which disciplines their behavior. The result is presented in quite a general setting, however, which necessitates the use of somewhat complicated sequences of money transfers to ensure that non-leaf players cannot misrepresent their information.[5]

I then apply this result to study when money is *essential*, in that the equilibrium payoff set is strictly larger with private cheap talk and money than with neither, or *strongly essential*, in that the equilibrium payoff set is strictly larger with private cheap talk and money than with cheap talk alone.[6] I show that a sufficient condition for money to be essential is that the network contains a "nice" subnetwork, which is a subtree in which every bilateral relationship has a product structure (Fudenberg and Levine, 1994) and in which there is some payoff vector that can be sustained in equilibrium with public monitoring that cannot be sustained in a "locally public equilibrium" (a generalization of perfect public equilibrium)

[4]For example, Wallace (2001) defines money as a "tangible useless object." See Section 6 for further discussion.

[5]Public information can also always be replicated with money alone: simply replace cheap talk messages with transfers of tiny amounts of money that nonetheless convey a large amount of information.

[6]Cf Lagos and Wright (2008), who write, "Money is said to be *essential* when the set of allocations that can be supported as equilibria is larger (or, sometimes, better) with money than without it." They attribute this notion to Hahn (1973).

with private monitoring.

Finally, I show that in two leading applications—repeated partnership games and favor trading games—the condition that the network contains a nice subnetwork reduces to the property that it contains a subtree of size at least three. Therefore, money is (strongly) essential in these models if the network contains a subtree of size at least three.

From an applied perspective, there are (at least) two ways of interpreting the results of the paper. The first is to take them at face value and conclude that in many models money can indeed be used to replicate public information and is therefore often essential. The second is to take them as a puzzle: what constraints on players' information or "rationality" must be added to the model to get a more realistic model of money? I hope that both interpretations may have value, and briefly discuss the second in the conclusion.

The paper proceeds as follows: Section 2 relates the paper to the literatures on repeated games, networks, and the microfoundations of money. Section 3 presents an example in which a simple and realistic way of using of money can expand the equilibrium payoff set, and argues that one must allow for more abstract and complicated uses of money to understand the limits of what money can do in networks. Section 4 then presents the model. Section 5 contains the benchmark results on replicating public information with cheap talk. Section 6 presents the main result on replicating public information with money. Section 7 shows how this result can be applied to show that money is essential. Section 8 applies the results of Sections 6 and 7 to partnership games, and Section 9 applies these results to favor trading games. Section 10 concludes. Proofs are contained in the appendix.

2 Related Literature

This paper is distinguished from the literature by its focus on when different communication technologies can replicate public information and on comparing cheap talk and money in a repeated game. Nonetheless, it is related to the literatures on community enforcement and evidence in repeated games, repeated games on networks, and the microfoundations of money.

The seminal paper on community enforcement and hard evidence in repeated games is Kandori (1992), who shows that cooperation is sustainable in the repeated prisoner's dilemma with random matching when players carry labels (such as "guilty" or "innocent") that are exogenously determined by their past play.[7] Most of the subsequent literature on community enforcement has not considered hard evidence per se: a recent exception is Fujiwara-Greve, Okuno-Fujiwara, and Suzuki (2012), who show that letting players issue "reference letters" can improve cooperation in a repeated prisoner's dilemma where players can either continue with their current partner or draw a new partner from the population. There is also a literature on the folk theorem in general private monitoring repeated games with communication, dating back to Compte (1998) and Kandori and Matsushima (1998). The main differences between my paper and this literature is that I restrict attention to repeated games on networks and compare the equilibrium payoff set with different communication technologies for a fixed discount factor rather than asking what communication or monitoring technology is needed for the folk theorem to hold.

The is also a rapidly growing literature on repeated games on networks. The folk theorems of Ben-Porath and Kahneman (1996) and Renault and Tomala (1998) are related to the benchmark results of Section 5 and are discussed there. Most of the rest of the literature studies more specific games. For example, Ahn and Suominen (2001) and Balmaceda and Escobar (2011) study how local communication among buyers can dissuade a seller from providing a low-quality good, and Lippert and Spagnolo (2011) and Ali and Miller (2012) study how local communication can help sustain cooperation in a repeated prisoner's dilemma.

Finally, this paper relates to the large literature on the microfoundations of money. I provide sufficient conditions for money to be essential in games with a finite, non-anonymous population of players interacting on a fixed network, relative to what could be achieved with cheap talk alone. In contrast, most of the existing literature considers games with a continuum of anonymous players interacting at random, and does not compare money with cheap talk; for example, this is the setting in the classic model of Kiyotaki and Wright (1993). There are some exceptions, however. Araujo (2004) adopts the arguments of Kandori and Ellison to show that money is essential in sufficiently large finite games with random

[7] This is closely related to the "norm equilibrium" of Okuno-Fujiwara and Postlewaite (1995). Milgrom, North and Weingast (1990) argue that "law merchants" in medieval Europe may have played a role in providing such labels. Ellison (1994) shows that cooperation is also sustainable without labels.

matching. Aliprantis, Camera, and Puzzello (2007) present a model with an infinite but non-anonymous population where money is essential even though players occasionally meet in centralized markets. Kocherlakota and Wallace (1998) show that money is essential with a continuum of players and random matching even in the presence of sufficiently unreliable public monitoring of individual actions.

A prominent paper in this literature that shares with mine the goal of comparing fiat money with other information technologies is Kocherlakota (1998). Kocherlakota's main result is that money is often inessential in the presence of a form of reliable public monitoring.[8] In contrast, I show that money is often essential when only private monitoring is available. Kocherlakota also gives an example in which money can replicate public monitoring and an example in which it cannot. In my model, money can replicate public monitoring quite generally; the primary reason for this difference is that players in my model are non-anonymous, which makes money—and repeated game effects more generally—much more powerful than in Kocherlakota's model.

3 A Simple Example

I begin by fleshing out the "three players on a line" example, which illustrates the need for the more general analysis of the rest of the paper.

To keep the example as simple as possible, suppose the players interact in continuous time. Players 1 and 2 are linked and players 2 and 3 are linked, but players 1 and 3 are not. If players i and j are linked, then at rate 1 player i gets the opportunity to do a favor for player j, and this opportunity is observed by i and j but not by the other player. Doing a favor costs c and yields benefit $b > c$ to the recipient, and the players have discount rate r. For this example only, I also let the players access a public randomizing device.

It is clear that, if the players are sufficiently patient—in particular, if $r \leq \frac{b-c}{c}$—they can sustain cooperation with "bilateral grim trigger strategies": if i fails to do a favor for j, then j never again does a favor for i. But if the players are less patient—if $r > \frac{b-c}{c}$—then it is

[8] More precisely, Kocherlakota's notion of *memory* is perfect information about one's partners' past play, their partners' past play, and so on.

shown in the appendix that no cooperation is possible (i.e., the unique sequential equilibrium payoff vector is 0). However, the appendix also shows that the players can sustain some cooperation for some $r > \frac{b-c}{c}$ if they have access to a single "dollar bill," by playing as follows:

Initially endow any player with the dollar. At any time, if player 2 (the middle player) has the dollar and player 1 or 3 gets the opportunity to do a favor for him, she does the favor in exchange for the dollar. If player 1 or 3 has the dollar, player 2 does a favor for her when he gets the opportunity, but when he does the favor he gets the dollar only with probability $\alpha^* = \frac{b+2c}{2b+c} < 1$. No one does favors for a player who does not have the dollar, and no one does favors for or transfers money to anyone who takes an off-path action.

This example has the appealing feature that the players use money in a realistic way: 1 and 3 only do favors in exchange for the dollar, and 2 only does favors in exchange for a chance of receiving the dollar.[9] But it also has several limitations: the game and the network structure are both very special, and the constructed equilibrium is still very inefficient. This illustrates a general tradeoff: as one requires that money is used more realistically, money becomes less powerful. This paper studies one extreme point on this frontier: while I maintain a standard *technological* definition of money—money is a tangible useless object, like in the example—I place no restrictions on the *strategic* role of money. For example, I do not require that players simply use money to "buy favors" as in the example. Studying what can be achieved with a restricted class of strategies—or in models where complicated strategies are infeasible, due to limits on players' information or rationality—is left as an intriguing direction for future research.

4 Model

This section describes the repeated game without communication and the notion of replicating public information. I add cheap talk to the model in Section 5 and add money in

[9] I conjecture that one could construct a very similar example where 2 does favors in exchange for a fraction of a dollar, so that the price of buying a favor is lower for 1 and 3 than for 2. The current example is simpler to analyze because the vector of money holdings takes only three values, given by the location of the dollar.

Section 6.

Players: There is a finite set of players $N = \{1, \ldots, n\}$ arranged on an undirected and connected network $L \subseteq P_2(N)$, the set of 2-element subsets of N, where $\{i, j\} \in L$ denotes a link between players i and j. The network will determine the structure of players' actions, payoffs, information, and—in subsequent sections—communication. In particular, it will become clear that the assumption that L is connected is essentially without loss of generality, as the fact that players only "interact" with their neighbors implies that if L is not connected one can replicate the analysis on each connected component of L. Let $N_i = \{j : \{i, j\} \in L\}$ be the set of player i's neighbors.

Stage game: Player i's stage-game action set is $A_i = \prod_{j \in N_i} A_{i,j}$, where the $A_{i,j}$ are arbitrary finite sets interpreted as player i's possible actions toward player j. There is a set of signal profiles $Z = \prod_{\{i,j\} \in L} Z_{i,j}$, where the $Z_{i,j} = Z_{j,i}$ are arbitrary finite sets interpreted as the signals that can be generated by the interaction between players i and j. It is assumed that there are probability distributions $\pi_{i,j}(\cdot | a_{i,j}, a_{j,i})$ such that the probability of signal $z_{i,j}$ conditional on action pair $(a_{i,j}, a_{j,i})$ is $\pi_{i,j}(z_{i,j} | a_{i,j}, a_{j,i})$, independent of the signal realizations for other pairs of players. That is, the probability of signal profile $z = (z_{i,j})_{\{i,j\} \in L}$ given action profile $a = (a_i)_{i \in N}$ is given by $\pi(z|a) = \prod_{\{i,j\} \in L} \pi_{i,j}(z_{i,j} | a_{i,j}, a_{j,i})$. Hence, the signal $z_{i,j}$ is "locally public," in that it is identically equal to $z_{j,i}$ but is completely uninformative about any other $z_{i',j'}$. Player i's stage-game expected payoff is $u_i(a) = \sum_{j \in N_i} \sum_{z_{i,j} \in Z_{i,j}} \pi_{i,j}(z_{i,j} | a_{i,j}, a_{j,i}) u^*_{i,j}(z_{i,j})$, where $u^*_{i,j} : Z_{i,j} \to \mathbb{R}$ gives player i's realized payoff from her interaction with player j. To save on notation, let $u_{i,j}(a_{i,j}, a_{j,i}) = \sum_{z_{i,j} \in Z_{i,j}} \pi_{i,j}(z_{i,j} | a_{i,j}, a_{j,i}) u^*_{i,j}(z_{i,j})$, and note that $u_i(a) = \sum_{j \in N_i} u_{i,j}(a_{i,j}, a_{j,i})$. Thus, $u_{i,j} : A_{i,j} \times A_{j,i} \to \mathbb{R}$ gives player i's expected payoff from her interaction with player j. For $\{i, j\} \in L$, I will refer to the two-player game $(A_{i,j}, A_{j,i}, Z_{i,j}, \pi_{i,j}, u_{i,j}, u_{j,i})$, which captures the direct relationship between i and j, as the (i, j)-*game*.

I assume throughout the paper that each (i, j)-game has a mutual-minmax Nash equilibrium. That is, I assume that every mixed action set $\Delta(A_{i,j})$ contains an element $\alpha^*_{i,j}$ such that the mixed action profile $\alpha^* = (\alpha^*_i)_{i \in L} = \left((\alpha^*_{i,j})_{j \in N_i} \right)_{i \in N}$ is a stage-game Nash equilibrium and

$$u_{i,j}\left(\alpha^*_{i,j},\alpha^*_{j,i}\right)=\min_{\alpha_{j,i}\in\Delta(A_{j,i})}\max_{\alpha_{i,j}\in\Delta(A_{i,j})}u_{i,j}\left(\alpha_{i,j},\alpha_{j,i}\right)\text{ for all }\{i,j\}\in L.$$

This assumption ensures that the worst possible punishments can be delivered "link by link," and thus do not require punishers to coordinate. It is needed for my results, because an outsider will generally be able to tell when a deviation occurs in the relationship between two players but will not be able to tell which one of them deviated.

Repeated game: The players play a repeated game in discrete time. At the beginning of period $t\in\{0,1,\ldots\}$, each player i chooses an action $a_{i,t}\in A_i$. The signal z_t is then drawn from $\pi(\cdot|a)$, payoffs are realized, and player i observes $(z_{i,j,t})_{j\in N_i}$.[10] Letting $h_{i,t}=\left(a_{i,t},(z_{i,j,t})_{j\in N_i}\right)$, player i's time-t history is $h_i^t=(h_{i,\tau})_{\tau=0}^{t-1}$ for $t\geq 1$, and every player has trivial initial history $h_i^0=h^0$. Letting H_i^t be the set of player i's time-t histories, a behavior strategy of player i's is a map $\sigma_i: H_i^t\to\Delta(A_i)$. Players have common discount factor $\delta\in(0,1)$. Denote the resulting repeated game by Γ_{PRI}, where the subscript PRI emphasizes that signal $z_{i,j}$ is private to the pair of players $\{i,j\}$ (though it is locally public between i and j). I will study the sequential equilibria (SE) of game Γ_{PRI}.[11] Let E_{PRI} be the set of SE payoffs of game Γ_{PRI}.

Replicating public information: Let Γ_{PUB} be the game in which the entire signal z is public. That is, Γ_{PUB} is derived from Γ_{PRI} by letting $h_{i,t}$ equal $(a_{i,t},z_t)$ rather than $\left(a_{i,t},(z_{i,j,t})_{j\in N_i}\right)$. Let E_{PUB} be the set of SE payoffs of game Γ_{PUB}. Below, I will define games Γ_{PRI}^{PUBCT}, Γ_{PRI}^{PRICT}, and $\Gamma_{PRI}^{\$}$ by adding public cheap talk, private (i.e., local) cheap talk, or money to the game Γ_{PRI}, and will denote the corresponding SE payoff sets by E_{PRI}^{PUBCT}, E_{PRI}^{PRICT}, and $E_{PRI}^{\$}$. I will say that public cheap talk (resp., private cheap talk, money) *can replicate public information* if $E_{PRI}^{PUBCT}\supseteq E_{PUB}$ (resp., $E_{PRI}^{PRICT}\supseteq E_{PUB}$, $E_{PRI}^{\$}\supseteq E_{PUB}$).[12] Informally, communication replicates public information if any payoff vector that can be attained in equilibrium when the local information is made public can also be attained with

[10] Thus, player i observes her own payoff.

[11] As usual, sequential equilibrium in repeated games with finite information sets in every period is defined by putting the product topology on the space of beliefs.

[12] There are games for which these inclusions are strict. I omit the proof of this fact, since it is not used in the paper.

communication.

5 Replicating Public Information with Cheap Talk

This section studies when public or private cheap talk may be used to replicate public information. The results of this section are broadly similar to results in the literature, and are included both for completeness and because there is a natural progression from these results to the main analysis of Section 6.

5.1 Public Cheap Talk

A game with public cheap talk $\Gamma_{PRI}^{PUBCT}(Y)$ is derived by augmenting the game Γ_{PRI} with a finite message set $Y = (Y_1, \ldots, Y_n)$ such that after players observe their private signals they simultaneously send public messages $y_i \in Y_i$. Formally, letting $h_{i,t} = \left(a_{i,t}, (z_{i,j,t})_{j \in N_i}, (y_{j,t})_{j \in N}\right)$, there are now two kinds of histories for every time t, denoted $h_i^{t-} = (h_{i,\tau})_{\tau=0}^{t-1}$ (called *action histories*) and $h_i^{t+} = \left((h_{i,\tau})_{\tau=0}^{t-1}, a_{i,t}, (z_{i,j,t})_{j \in N_i}\right)$ (called *communication histories*), and a strategy maps action histories to $\Delta(A_i)$ and maps communication histories to $\Delta(Y_i)$. Let $E_{PRI}^{PUBCT}(Y)$ be the SE payoff set of $\Gamma_{PRI}^{PUBCT}(Y)$, and let $E_{PRI}^{PUBCT} = \bigcup_Y E_{PRI}^{PUBCT}(Y)$, where the union is taken over all finite sets Y.

The first benchmark result is that public cheap talk can always replicate public information. The proof is very simple: Take a SE σ^{PUB} in the public monitoring game Γ_{PUB}. Specify that after every round of play the players publicly report what signals they observe and then play according to σ^{PUB}, taking the reported signals as the true ones. If any reports disagree, play the mutual-minmax profile α^* forever. This strategy profile yields the same payoffs as σ^{PUB}, and it can be shown to be a SE.[13]

Theorem A Public cheap talk can replicate public information (i.e., $E_{PRI}^{PUBCT} \supseteq E_{PUB}$).

[13]This argument clearly relies heavily on the assumption that signal $z_{i,j}$ is locally public between i and j. If signals were not even locally public, then one would be in the setting of general repeated games with private monitoring, and public information could not be replicated even with public communication. However, it may be the case that if signals are "almost locally public" then public communication can "almost replicate public information." I do not pursue this question here.

Theorem A is related to Theorem 1 of Ben-Porath and Kahneman (1996), which establishes the folk theorem for repeated games with public communication where each player is perfectly observed by at least two others. Here, it is enough that i and j observe the same $z_{i,j}$, because there is a mutual-minmax Nash equilibrium. Also, Theorem A is not a folk theorem but rather a result about replicating public information for fixed δ.

5.2 Private Cheap Talk

In this paper, "private cheap talk" means communication along the links of the network. That is, with private cheap talk players can communicate directly with their neighbors but not with other players. However, I allow multiple rounds of communication after every round of play, which lets players communicate indirectly with players to whom they are not linked.[14]

A game with private cheap talk $\Gamma_{PRI}^{PRICT}(Y)$ is derived by augmenting the game Γ_{PRI} with a finite message set $Y = (Y_{i,j})_{\{i,j\} \in L}$ such that after players observe their private signals they have unboundedly many opportunities to simultaneously send private messages $y_{i,j}^k \in Y_i$, where the subscript denotes a message from i to j and the superscript $k \in \mathbb{N}$ denotes the number of the communication round. To allow for unboundedly many rounds of communication while keeping the formal definition of strategies as simple as possible, I will assume that in each round each player i simultaneously choose whether to send a message in $Y_{i,j}$ to each player $j \in N_i$ or to send "no message" to player j, denoted $\emptyset_{i,j}$. If any player sends a message in $Y_{i,j}$ to another player in round k, then another round of communication occurs. If all players send "no message" to all of their neighbors, then the communication phase ends and the game moves to the next round of play. If players send messages forever, so that

[14] In any communication round, a player may learning something that she would like to pass on. To accomodate this, I allow for unboundedly many rounds of communication after each round of play.

play never moves to the next round of play, I specify that all players receive payoff $-\infty$.[15,16]

Formally, let K_t be the number of communication rounds in period t (which is determined endogenously), and let $h_{i,t} = \left(a_{i,t}, (z_{i,j,t})_{j\in N_i}, \left(y^k_{i,j,t}, y^k_{j,i,t}\right)_{j\in N_i, k\in\{1,\ldots,K_t\}}\right)$. There are now unboundedly many kinds of histories for every period t, denoted $h_i^{t-} = (h_{i,\tau})_{\tau=0}^{t-1}$ (action histories), $h_i^{t,0} = \left((h_{i,\tau})_{\tau=0}^{t-1}, a_{i,t}, (z_{i,j,t})_{j\in N_i}\right)$, and $h_i^{t,k} = \left((h_{i,\tau})_{\tau=0}^{t-1}, a_{i,t}, (z_{i,j,t})_{j\in N_i}, \left(y^{k'}_{i,j,t}, y^{k'}_{j,i,t}\right)_{j\in N_i, k'\in\{1,\ldots,k\}}\right)$, for $k = \{1,\ldots,K_t\}$ (communication histories). A strategy σ_i now maps action histories h_i^{t-} into $\Delta(A_i)$ and maps communication histories $h_i^{t,k}$ to $\Delta\left((Y_{i,j}\cup\{\emptyset_{i,j}\})_{j\in N_i}\right)$. Let $E^{PRICT}_{PRI}(Y)$ be the SE payoff set of $\Gamma^{PRICT}_{PRI}(Y)$, and let $E^{PRICT}_{PRI} = \bigcup_Y E^{PRICT}_{PRI}(Y)$.

The second benchmark result is that private cheap talk can replicate public information if the network L is 2-connected. Recall that a network is 2-connected if there are at least two independent paths (i.e., two paths with disjoint sets of internal nodes) between every pair of nodes. The main idea is again quite simple: Start with a SE σ^{PUB} in game Γ_{PUB}. Specify that after every round of play there are several rounds of communication in which players report both the signals they have observed directly and the signals that have been reported to them in earlier rounds, until all signals have been reported to all players. The players then play according to σ^{PUB}, taking the reported signals as the true ones. If a player sends or receives an inconsistent report, she then reports that there has been a deviation, and the news of the deviation spreads throughout the network and leads all players to play the mutual-minmax profile α^*. The assumption that the network is 2-connected implies that no player can deceive another about the signals: if a player i lies about a signal to one of her neighbors, the neighbor will eventually receive a conflicting report via a path that does not include i, and will then revert to α^*. See the proof for details of off-path play and

[15] There are several ways in which one could choose to model the possibility of unboundedly many rounds of communication. The advantage of the approach taken here is its simplicity. An unappealing feature of the current approach is that it does not specify how the players know when everyone has sent "no message," i.e., how they know when it is time to move to the next play phase. An alternative model that might be more appealing in this dimension is specifying that there are infinitely many rounds of communication between each round of play (rather than an unbounded but finite number), as in Aumann and Hart (2003). I conjecture that such a model would yield the same conclusions as the current model, but defining strategies in such a model is considerably more complicated.

[16] One might be concerned that that this lets players provide incentives by threatening to play the equilibrium where all players send messages forever and all receive payoff $-\infty$. The possibility can be ruled out by, for example, letting players send a special message $\emptyset_i^*$ that binds them to send $\emptyset_{i,j}$ to all their neighbors for the rest of the communication phase. The larger point is that the specification of payoffs when players communicate forever plays no substantive role and serves only to ensure that the game is well-defined.

verification that the resulting strategy profile is a SE profile.

Theorem B Private cheap talk can replicate public information (i.e., $E_{PRI}^{PRICT} \supseteq E_{PUB}$) if the network L is 2-connected.

Theorem B is related to Theorem 2.6 of Renault and Tomala (1998), which gives a Nash folk theorem for repeated games with a 2-connected monitoring network without communication. Theorem B avoids some complications that emerge in their paper by allowing communication and assuming a mutual-minmax Nash equilibrium (though Theorem B is for sequential equilibrium rather than Nash). Moreover, the results differ in that Theorem B is about replicating public information for fixed δ.

6 Replicating Public Information with Money

I now turn to the main part of the analysis, where players have access to fiat money.

A game with money $\Gamma_{PRI}^{\$}$ is similar to a game with private cheap talk, except that in addition to sending cheap talk messages players can also transfer quantities of money to each other.[17] The difference between cheap talk and money is that a player can send any cheap talk message she wants, but can only send money that she is currently holding: for example, any player can say "message number 5," but only a player with at least 5 dollars can make a $5 transfer. This is quite consistent with the modern view of what money is. For example, Wallace (2001) differentiates money from an arbitrary state variable as follows:

> "... misrepresentation of holdings of a tangible object is limited by the possibility that others can at least say 'show me.' For an intangible state variable, there are no limits on misrepresentation if there is no monitoring."

Formally, a game with money $\Gamma_{PRI}^{\$}(Y, m^0)$ is derived from the game with private cheap talk $\Gamma_{PRI}^{PRICT}(Y)$ by specifying an initial vector of money holdings $m^0 = (m_1^0, \ldots, m_n^0)$, with $m_i^0 \in \mathbb{Q}_+$ for all $i \in N$, where $\mathbb{Q}_+$ denotes the non-negative rationals, and allowing players to

[17] As indicated in the introduction, it would be essentially equivalent to let players transfer only money and not also cheap talk messages. The current approach is slightly easier to exposit.

transfer money concurrently with their messages. That is, at every history in $\Gamma_{PRI}^{PRICT}(Y)$ where player i chooses a message $y_{i,j} \in Y_{i,j}$ to send to player j, she now chooses a pair $(y_{i,j}, m_{i,j}) \in Y_{i,j} \times \mathbb{Q}_+$ to send to player j, subject to the constraint that $\sum_{j \in N_i} m_{i,j} \leq m_i$, where m_i is player i's current money holding, and the vector of money holding is then updated to

$$m_i' = m_i + \sum_{j \in N_i} (m_{j,i} - m_{i,j}).$$

To be clear, in each communication phase there is another round of communication if any player sends a message or transfers money. The point of allowing players to transfer only rational quantities of money is to ensure that the strategy space remains countable, so that the standard definition of sequential equilibrium continues to apply.

The following is the main result of the paper. Here, $E_{PRI}^{\$} = \bigcup_{(Y,m^0)} E_{PRI}^{\$}(Y, m^0)$, where $E_{PRI}^{\$}(Y, m^0)$ is the SE payoff set in $\Gamma_{PRI}^{\$}(Y, m^0)$.

Theorem 1 *Money can replicate public information (i.e., $E_{PRI}^{\$} \supseteq E_{PUB}$).*

Theorem 1 does not require the strong assumption that the network is 2-connected. The fact that money can replicate public information even when the network is not 2-connected will form the basis of the later results on the essentiality of money (Theorems 2-4).

To see the overall approach of the proof, suppose that L is a tree. Let σ^{PUB} be a SE strategy profile in game Γ_{PUB}. Initially, endow each of the "leaf players" in L (i.e., players with only one neighbor) with a large amount of money, and endow "non-leaf players" with no money. Have players initially play as in σ^{PUB}. After each round of play, first have players repeatedly report their signals to each other as in the model with private cheap talk: this is called the "reporting subphase" in the proof. Then have players use money to check that no players have misreported signals: this is called the "confirmation subphase" and is described below. In the next round, players play according to σ^{PUB}, taking the reported signals as the true ones.

As in Theorems A and B, if players can be induced to report their signals truthfully, the above construction yields an equilibrium with the same payoffs as σ^{PUB}. Thus, the key insight behind Theorem 1 is that the confirmation subphase can be constructed so as to ensure that no player can mislead another about the value of any signal. The construction is as follows: Assign a natural number k to every possible vector of signals z. At the beginning of the period t confirmation subphase, each leaf player i thinks that the true vector of period t signals is some $\hat{z}^i$. The confirmation subphase starts with some leaf player—say, player 1—sending k_1 dollars down the path toward another leaf player—call him player 2—where k_1 is the number assigned to $\hat{z}^1$. The non-leaf players on this path are supposed to pass the money on to player 2. When player 2 receives the transfer, he checks whether it equals the number assigned to $\hat{z}^2$, the vector of reports he has received. If it does, he adds an additional k_2 dollars to the transfer he received, and passes this new larger "pot" of money on to the next leaf player, player 3. This process continues until each leaf player get the chance to add money to the pot, and the pot is then returned to player 1. Finally, player 1 then makes an additional large transfer to each leaf player, who then returns this transfer to player 1.

While it is hard to give a complete intuition for why this construction works without just proving the theorem, it is worth noting the following three important facts. First, if a non-leaf player fails to pass the correct amount of money on toward the next leaf player, then player 1 does not get back a pot of the correct size, and thus does not make the final confirmatory transfers to the other leaf players (who can then infer that a deviation occurred). Second, if any leaf players disagree as to the vector of period t signals, then player 1 again does not get back a pot of the correct size. Finally, the transfers can be constructed so that a player can never save enough money in an earlier round to mislead another player in a later round. Broadly speaking, the construction ensures that no player can "afford" to deviate in a profitable way.[18]

I conclude this section with two remarks on Theorem 1. First, the choice of m^0 is not crucial. As a consequence, even if one takes the view that initial money holdings are

[18] Another issue in the proof of Theorem 1 is that explicitly describing off-path play is intractable. This is because when player i sees player j deviate, player i may not be willing to pass on this information if doing so leads all players to play the mutual-minmax profile α^*. To address this, I specify only that when player i observes a deviation by player j, she minmaxes player j himself as well as those other players who lie on a path from i to j in L. The rest of off-path play is left unspecified, and I appeal to Kakutani's fixed point theorem to show that a sequential equilibrium σ^{PRI} with the desired on-path properties exists. For related non-constructive approaches to specifying off-path play in repeated games with private information, see Hörner and Olszewski (2006) or Escobar and Toikka (2012).

exogenously given, public information can still be replicated for a wide range of initial money holdings. The idea is that if any non-leaf players are endowed with money, they can in effect be forced to transfer all of their money to player 1 at the beginning of the game. Formally, one can show the following result:[19]

Proposition 1 *Suppose the initial vector of money holdings m^0 is exogenously given. If there exists a spanning tree $L' \subseteq L$ such that $m_i^0 > 0$ for all leaf players i in L', then $E_{PRI}^{\$}(Y, m^0) \supseteq E_{PUB}$, where Y is the message set from the proof of Theorem 1.*

Second, Theorem 1 relies on the assumption that money is infinitely divisible. This assumption serves two roles in the proof. First, it lets one ensure that leaf players never run out of money. This could potentially be addressed by instead "rebalancing" money holdings between rounds, although it is not completely clear how to do this. Second—and more importantly—it allows the size of the final confirmation transfer in each period to increase over time. This ensures that a player who deviates by saving some money cannot use it to mimic a later confirmation transfer. Both of these roles of infinite divisibility could instead be filled by simply disbursing more money to the leaf players every period, if this were allowed (contrary to my assumptions). For example, Theorem 1 would go through if dollars are indivisible but $\$|Z|$ are disbursed from a bank to each leaf player in every period. I do not know if Theorem 1 holds with indivisible dollars without allowing such exogenous transfers.

7 From Replication to Essentiality

Theorem 1 itself, which shows that money facilitates replicating public information in repeated games on networks, is one of the two main contributions of this paper. The second main contribution is using Theorem 1 to show that money is essential—in that the SE payoff set is larger with money than without it—in a broad class of games. I use the following definition:

[19] A *spanning tree* is a connected subnetwork with no cycles that contains all the nodes in the original network.

Definition 1 *Money is* essential *if* $E^{\$}_{PRI} \supsetneq E_{PRI}$.

Money is strongly essential *if* $E^{\$}_{PRI} \supsetneq E^{PRICT}_{PRI}$.

The latter property is indeed stronger because $E^{PRICT}_{PRI}(Y) \supseteq E_{PRI}$ for every message set Y, as messages can always be ignored.

How can one tell whether money is essential in a particular game? A first observation is that $E^{\$}_{PRI} \supseteq E_{PRI}$ and $E^{\$}_{PRI} \supseteq E^{PRICT}_{PRI}$ are trivially true: any SE is Γ_{PRI} or Γ^{PRICT}_{PRI} can be turned into a payoff-equivalent SE in $\Gamma^{\$}_{PRI}$ by specifying that players never make transfers and ignore transfers if they are made (in particular, $E^{\$}_{PRI}(Y, m^0) \supseteq E_{PRI}$ and $E^{\$}_{PRI}(Y, m^0) \supseteq E^{PRICT}_{PRI}(Y)$ for any (Y, m^0)). Combining this observation with Theorem 1 yields the following corollary:

Corollary 1 *Money is essential if $E_{PUB} \backslash E_{PRI} \neq \emptyset$. Money is strongly essential if $E_{PUB} \backslash E^{PRICT}_{PRI} \neq \emptyset$.*

Proof. By Theorem 1, $E^{\$}_{PRI} \supseteq E_{PUB}$. So $E_{PUB} \backslash E_{PRI} \neq \emptyset$ (resp., $E_{PUB} \backslash E^{PRICT}_{PRI} \neq \emptyset$) implies that $E^{\$}_{PRI} \backslash E_{PRI} \neq \emptyset$ (resp., $E^{\$}_{PRI} \backslash E^{PRICT}_{PRI} \neq \emptyset$). The observation that $E^{\$}_{PRI} \supseteq E_{PRI}$ completes the proof for "essential," and the observation that $E^{\$}_{PRI} \supseteq E^{PRICT}_{PRI}$ completes the proof for "strongly essential." ■

While Corollary 1 is quite general, it is not a very useful tool for determining when money is essential because it can be hard to know when $E_{PUB} \backslash E_{PRI} \neq \emptyset$ or $E_{PUB} \backslash E^{PRICT}_{PRI} \neq \emptyset$. The difficulty is that the set E_{PRI} of all sequential equilibrium payoffs in the private monitoring game Γ_{PRI} is usually impossible to characterize for fixed discount factors, as is the set E^{PRICT}_{PRI}. However, I will show that one can often establish that money is essential or strongly essential while restricting attention to the following much more tractable class of strategies:

Definition 2 *A* locally public strategy σ_i *is a strategy in* Γ_{PRI} *where* $\sigma_{i,j}$ *depends only on* $(z_{i,j,\tau})_{\tau=0}^{t-1}$, *for all* $j \in N_i$. *A* locally public equilibrium *(LPE) in* Γ_{PRI} *is a SE in* Γ_{PRI} *in locally public strategies. Denote the LPE payoff set in* Γ_{PRI} *by* E^{LPE}_{PRI}.

A local cheap talk strategy σ_i *is a strategy in* Γ^{PRICT}_{PRI} *where* $\sigma_{i,j}$ *depends only on* $\left\{z_{i,j,\tau}, \left(y^k_{i,j,\tau}, y^k_{j,i,\tau}\right)_{k \in \{1,\ldots,K_\tau\}}\right\}_{\tau=0}^{t-1}$, *for all* $j \in N_i$. *A* local cheap talk equilibrium *(LCTE)* *in Γ_{PRI}^{PRICT} is a SE in Γ_{PRI}^{PRICT} in local cheap talk strategies. Denote the LCTE payoff set in Γ_{PRI}^{PRICT} by E_{PRI}^{LCTE}.*

Thus a locally public strategy is one where player i conditions her play in her relationship with player j only on the history of locally public signals between i and j, and a local cheap talk strategy is one where player i conditions her play in her relationship with player j (including the messages she sends to j) only on the history of locally public signals and cheap talk between i and j. Locally public equilibrium is the natural analog of the standard public perfect equilibrium (PPE) in repeated games with imperfect public monitoring, and local cheap talk equilibrium is the natural analog of PPE when players can send messages only about mutually public information. Note that with local cheap talk strategies players have very little to talk about, since they do not condition their messages on information that the receiver does not already have. In particular, the set of LCTE payoffs is essentially the set of LPE payoffs in the auxiliary game where each pair of players is given access to a public randomizing device.[20]

I now show that the condition that $E_{PUB} \backslash E_{PRI} \neq \emptyset$ (resp., $E_{PUB} \backslash E_{PRI}^{PRICT} \neq \emptyset$) in Corollary 1 may be replaced with something like $E_{PUB} \backslash E_{PRI}^{LPE} \neq \emptyset$ (resp., $E_{PUB} \backslash E_{PRI}^{LCTE} \neq \emptyset$). To do this, I introduce the notion of a "nice" subnetwork.

For any subnetwork $M \subseteq L$, let $E|_M$ be the SE payoff set in the game where M is the original network, or equivalently the SE payoff set in the game where all links $\{i, j\} \notin M$ are deleted (so that $E_{PRI}|_M$ is the SE payoff set in this game with private monitoring, $E_{PUB}|_M$ is the SE payoff set in this game with public monitoring, etc.). For future reference, the game where M is the original network will be denoted $\Gamma|_M$. Finally, for any set X, let $\text{co}(X)$ denote the convex hull of X. I now introduce a key definition:[21]

Definition 3 *A subnetwork $M \subseteq L$ is* nice *if it has the following three properties:*

1. M is a subtree of L. That is, for any players $i, j \in M$, there is a unique path from i

[20] I say "essentially" here because I do not prove a formal result to this effect and there would be some technicalities arising from the requirement that the message set is finite.

[21] I slightly abuse notation here by letting M stand for both a subnetwork of L and the set of nodes in that subnetwork.

to j in L, and every node in this path is contained in M.[22]

2. *For all $\{i,j\} \in M$, the (i,j)-game has a product structure. That is, $Z_{i,j} = Z_{i,j}^i \times Z_{i,j}^j$ and $\pi_{i,j}\left(z_{i,j}|a_{i,j}, a_{j,i}\right) = \pi_{i,j}^i\left(z_{i,j}^i|a_{i,j}\right)\pi_{i,j}^j\left(z_{i,j}^j|a_{j,i}\right)$.*

3. $E_{PUB}|_M \backslash \operatorname{co}\left(E_{PRI}^{LPE}|_M\right) \neq \emptyset$.

In addition, M is truly nice *if the last condition can be strengthened to $E_{PUB}|_M \backslash \operatorname{co}\left(E_{PRI}^{LCTE}|_M\right) \neq \emptyset$.*

The following theorem is the key tool for determining when money is essential.

Theorem 2 *Money is essential if L contains a nice subnetwork. Money is strongly essential if L contains a truly nice subnetwork.*

For example, if L is a tree, all (i,j)-games in L have a product structure, and $E_{PUB} \backslash \operatorname{co}\left(E_{PRI}^{LPE}\right) \neq \emptyset$, then Theorem 2 says that money is essential.[23] However, Theorem 2 is much more general that this because L itself need not be nice. In particular, the condition that L is a tree is very strong, but the condition that L contains a subtree is trivial. However, not any subtree will do: in particular, if every (i,j)-game in M has a product structure, then $E_{PUB}|_M \backslash \operatorname{co}\left(E_{PRI}^{LPE}|_M\right) \neq \emptyset$ can hold only if M contains at least three players. In the applications of Sections 8 and 9, the approach will be to show that any subtree of size at least three is truly nice, and conclude that money is strongly essential whenever L contains a subtree of size at least three.

The intuition for Theorem 2 is as follows: If M is a tree and all (i,j)-games in M have a product structure, then in the game where M is the original network it is without loss of generality to restrict attention to LPE. If in addition M is a subtree of L, then the equilibrium payoff set on L equals the sum of the equilibrium payoff set on M and the equilibrium payoff set on $L \backslash M$ (in Γ_{PRI}). So if money expands the equilibrium payoff set on M while restricting attention to LPE, then it also expands the unrestricted equilibrium payoff set on L.

[22] This is stronger than the condition that M is itself a tree: it is not enough that there is a unique path from i to j in M.

[23] Technically, one can show that $E_{PUB} \backslash E_{PRI}^{LPE} \neq \emptyset$ is sufficient here.

8 Application 1: Partnership with Perfect Local Monitoring

This section shows that money is strongly essential in a standard repeated partnership game/prisoner's dilemma on a network, as long as players are patient enough to sustain some effort in equilibrium and the network contains a subtree of size at least three.

Assume that linked players each choose a real-valued action for their relationship—interpreted as their "effort levels"—and observe each other's action: formally, $A_{i,j}$ is a finite subset of $\mathbb{R}_+$ containing 0, with $A_{i,j} = A_{i',j'}$ for $\{i,j\}, \{i',j'\} \in L$, $Z_{i,j} = A_{i,j} \times A_{j,i}$, and $\pi_{i,j}\left((a_{i,j}, a_{j,i}) | a_{i,j}, a_{j,i}\right) = 1$. Assume also that

$$u_{i,j}\left(a_{i,j}, a_{j,i}\right) = \frac{1}{2} f\left(a_{i,j}, a_{j,i}\right) - a_{i,j},$$

where $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a symmetric joint production function satisfying $f(0,0) = 0$, $f_i \in (0,2)$, $f_i(0,0) > 1$, and f concave, where the subscripts here denote partial derivatives.[24] I also assume that there exist $\hat{a}_{i,j} > 0$ such that $f(\hat{a}_{i,j}, \hat{a}_{i,j}) - \hat{a}_{i,j} = 0$. Note that this game is a prisoner's dilemma, in that u_i is decreasing in each $a_{i,j}$ and increasing in each $a_{j,i}$, and that setting $a_{i,j} = 0$ for all $j \in N_i$ is dominant for player i in the stage game. In particular, $a = 0$ is a mutual-minmax Nash equilibrium.

The assumption that $A_{i,j}$ is finite is necessary to apply Theorem 2.[25] However, since essentiality requires that money *strictly* expands the SE payoff set, I will need to assume that $A_{i,j}$ is "sufficiently dense"; for example, if $A_{i,j} = \{0, 1000000\}$, then the unique SE payoff vector may be 0 with or without money. To rule this out, I say that $A_{i,j}$ is ε-*dense* if for all $a_{i,j} \in [0, \hat{a}_{i,j}]$ there exists $a'_{i,j} \in A_{i,j}$ such that $\left|a_{i,j} - a'_{i,j}\right| < \varepsilon$, and I will assume that $A_{i,j}$ is ε-dense for small ε.

The result is the following:

Theorem 3 *In the partnership game, there exists $\varepsilon > 0$ such that if $A_{i,j}$ is ε-dense, $\delta > \frac{2 - f_i(0,0)}{f_i(0,0)}$, and L contains a subtree of size at least three, then money is strongly essential.*

[24] The assumption that f is symmetric serves only to simplify notation.

[25] The only place where finiteness of $A_{i,j}$ is used in Theorems 1 and 2 is the appeal to existence of sequential equilibrium in finite games in the proof of Theorem 1.

The proof (in the Appendix) proceeds by showing that any subtree $M = \{\{1,2\},\{2,3\}\}$—which must exist if L contains a subtree of size at least three—is truly nice, and then applying Theorem 2. The intuition for why such a subtree is truly nice is that with money one can specify that the equilibrium in the $(2,3)$ game is favorable to player 2 and that players 2 and 3 stop cooperating if 2 fails to cooperate with 1. This creates slack in player 2's incentive constraint in the $(1,2)$ game, which allows equilibrium in the $(1,2)$ game to be more favorable for 1 than is any equilibrium without money.[26] Broadly speaking, in this application money expands the equilibrium payoff set by allowing the pooling of incentive constraints across relationships.

9 Application 2: Trading Favors

This section shows that money is strongly essential in a networked version of a standard continuous-time trading favors game (Möbius, 2000; Hauser and Hopenhayn, 2010), as long as the network contains a subtree of size at least three.

The game Γ_{PRI} is now as follows: Time runs continuously from 0 to ∞. If $\{i,j\} \in L$, then at rate 1 player i gets the opportunity to do a favor for player j, independently across $i, j \in N$. Only i observes when she has an opportunity to do a favor for j, though both i and j see when i actually does a favor for j. Doing a favor costs $c > 0$, and receiving a favor gives benefit $b > c$. The players have common discount rate $r > 0$. Thus, if S_i is the set of times at which player i does a favor and T_i is the set of times at which player i receives a favor, then player i's payoff is

$$\left(\sum_{t \in T_i} e^{-rt} b\right) - \left(\sum_{t \in S_i} e^{-rt} c\right).$$

I also let the players observe the outcome of a public randomizing device $\omega_t \in [0,1]$ at every time t.[27]

[26] The assumption that $\delta > \frac{2 - f_i(0,0)}{f_i(0,0)}$ ensures that some effort can be sustained in a bilateral relationship, which is clearly necessary for this intuition to apply.

[27] The public randomizing device could be dispensed with by having the players play jointly controlled lotteries in the communication phase (e.g., Aumann and Hart, 2003). Its only purpose is to minimize the number of the changes that have to be made to the communication phase strategies from the proof of

The game Γ_{PUB} is identical to Γ_{PRI}, except that now all players observe when i does a favor for j.

Finally, the game $\Gamma^{\$}_{PRI}$ is identical to Γ_{PRI}, except that now, after i gets an opportunity to do a favor for j (but before the realization of the public randomizing device ω_t), there is a phase of communication and transfers, described below. That is, the timing is:

1. Opportunities to do favors are determined, and are observed by the players who can do them.

2. Communication takes place.

3. Players observe the outcome of the public randomizing device.

4. Players with the opportunity to do favors choose whether or not to do them.

In the communication phase, players other than i learn that a communication phase has begun only if they receive a report, message, or transfer from a player who received one from a player who received one from… player i. Formally, if i gets the opportunity to do a favor for j at time t, then the period t communication phase is exactly as in Section 6, with the modification that in round 1 only i is allowed to send messages or transfers to their neighbors, and in subsequent round only players who have received a message or transfer at some earlier round (in addition to i herself) are allowed to send messages or transfers.[28] As before, the phase ends when no one sends a message or transfer.

The goal of this section is to use Theorems 1 and 2 to show that money is strongly essential in the trading favors game. However, the trading favors game does not formally fit into the model of Section 3 (e.g., it is not a repeated game), so Theorems 1 and 2 cannot be immediately applied. The following proposition adapts Theorems 1 and 2 to the trading favors game:[29]

Theorem 1.

[28] There is also a difference in notational convention: in Section 6, the period t communication phase occurs after the period t action phase, whereas here the time t communication phase occurs before players decide whether or not to do favors at time t.

[29] Note that the requirement that every $\{i, j\}$ game in M has a product structure does not appear in Proposition 2. This is because the property of repeated games with a product structure that is needed for Theorem 2 is that player i's beliefs about player j's private history conditional on player i's actions and the public history between i and j equals her beliefs about player j's private history conditional only on the public history, and this property holds in the trading favors game.

Proposition 2 *In the trading favors game, money is strongly essential if L contains a subtree M with $E_{PUB}|_M \backslash \operatorname{co}\left(E_{PRI}^{LCTE}|_M\right) \neq \emptyset$.*

The main result of this section is the following:

Theorem 4 *In the trading favors game with unobservable opportunities, if $r \leq \frac{b-c}{c}$ and L contains a subtree of size at least three, then money is strongly essential.*

The proof (in the Appendix) again shows that any subtree $M = \{(1,2),(2,3)\}$ is truly nice and applies Proposition 2. The proof that any such subtree is truly nice uses the differential characterization of the PPE payoff set in the two-player trading favors game with unobservable opportunities due to Hauser and Hopenhayn (2010) (which in turn is an application of the results of Abreu, Pearce, and Stacchetti (1990)), and also requires some new results for that game. The intuition is that the source of inefficiency in the two-player game comes from "hitting the individual rationality constraints": at some point, player 2 cannot do more favors for player 1, because player 1 would then "owe" so many favors that she would prefer to revert to autarky. In the three-player game with public monitoring—or money—one can specify that when player 2 does a favor for player 1, some of the "repayment" for the favor has to come from player 3. Roughly speaking, player 1 and 3's individual rationality constraints can be partially pooled, which reduces inefficiency.

Finally, note that money is also essential—for an open interval of discount rates—in the model where both i and j can observe when i has an opportunity to do a favor for j. This is precisely the example of Section 3.

10 Conclusion

This paper has compared cheap talk and fiat money as means of replicating public information in repeated games on networks. The main result is that public information can always be replicated when money is available—in contrast, it can only be replicated when the network is 2-connected if money is unavailable. This leads to a simple sufficient condition for the essentiality of money: money is essential if the network contains a nice subnetwork, i.e., a subtree on which replicating public information may be shown to be valuable while restricting attention to locally public equilibria. In leading applications, this condition reduces to the property that the network contains a subtree of size at least three.

As I have mentioned above, an important direction for future research is to impose limits on players' information or rationality and see what this implies for what payoffs can be achieved with money and for how money is used. For example, it would be interesting to see how the results of this paper extend to the case where the network is uncertain: recalling the example of three players in a line, if 1 and 3 do not have a relationship, then assuming that each knows that the other has a relationship with 2 seems strong. Similarly, it would be interesting to see how the results extend when the initial distribution of money is unknown. Other promising directions include studying models where players are more anonymous than in this paper but less anonymous than in standard continuum agent–random matching models of money, or models where players use maxmin optimal strategies or other "boundedly rational" rules in the face of uncertainty about the distribution of money.

11 Appendix

11.1 Details for Section 3

I first show that no cooperation is possible without money if $r > \frac{b-c}{c}$ (even if private cheap talk is allowed), and then show that the proposed strategy profile with money is an equilibrium if $r \leq \frac{b-\alpha^* c}{c}$.

For the game without money, a simple extension of the proof of Theorem 2 (which may be found in an earlier version of the paper) shows that $E_{PRI}^{PRICT} = E_{PRI}^{LCTE}$. Let $\bar{a}_i$ be player i's expected discounted future action toward player $j \neq i$, for $i, j \in \{1, 2\}$:

$$\bar{a}_i \equiv r \int_{s=t}^{\infty} e^{-r(s-t)} E\left[a_{i,s} \middle| \left(z_{i,j,\tau}\right)_{\tau=0}^{t}\right] ds,$$

where $a_{i,t} \in [0, 1]$ is the probability that player i does a favor for player j when she gets the opportunity to do so at time t.[30] So when player i is supposed to do a favor for player j at

[30] I impose here that $a_{i,t}$ is measurable over intervals where no favors are done, which implies that the

time t, her payoff in the (i,j)-game from conforming is $\frac{1}{r}(b\bar{a}_j - c\bar{a}_i) - c$, and her payoff in the (i,j)-game from deviating to playing $a_i = 0$ forever is 0. Hence, in any LCTE in which $\bar{a}_i > 0$,

$$r \leq \frac{b\bar{a}_j - c\bar{a}_i}{c}.$$

So in any such LCTE it must also be that $\bar{a}_j > 0$. Suppose, toward a contradiction, that such an LCTE exists. Then without loss of generality $\bar{a}_i \geq \bar{a}_j$. So it must be that

$$r \leq \left(\frac{b-c}{c}\right)\bar{a}_i.$$

Since $\bar{a}_i \in [0,1]$ and $r > \frac{b-c}{c}$, this is impossible. So there is no LCTE with $\bar{a}_i > 0$ for any i, so $E_{PRI}^{LCTE} = E_{PRI}^{PRICT}$ is just the 0 vector.

For the strategy profile with money, let V_i be player 2's continuation value when player i has the dollar. Noting that $V_1 = V_3$, we have

$$\begin{aligned} rV_2 &= 2(V_1 - V_2 + b), \\ rV_1 &= \alpha(V_2 - V_1) - c. \end{aligned} \tag{1}$$

Player 2's strategy if optimal if and only if

$$\alpha V_2 + (1-\alpha)V_1 \geq c,$$

since if he does a favor he gets V_2 with probability α and gets V_1 with probability $1-\alpha$, and if he fails to do a favor he gets 0 (as he never again gets the dollar). By (1), this inequality holds if and only if

$$V_1 \geq 0,$$

or equivalently

$$V_2 \geq c.$$

integral is well-defined.

Some algebra shows that this is the case if and only if

$$\alpha \geq \frac{c}{b}\left(1+r/2\right).$$

Next, let W_i be player 1's continuation value when player i has the dollar. Then

$$\begin{aligned} rW_1 &= \alpha\left(W_2 - W_1\right) + b \\ rW_2 &= W_1 - W_2 - c + W_3 - W_2 \qquad (2) \\ rW_3 &= \alpha\left(W_2 - W_3\right). \qquad (3) \end{aligned}$$

Player 1's strategy is optimal if and only if

$$W_1 \geq c,$$

since if she does a favor she gets W_1 and if she fails to do a favor she gets 0. By (2) and (3), this holds if and only if

$$W_2 \geq 0.$$

Some algebra shows that this is the case if and only if

$$\alpha \leq \frac{b}{c} - r.$$

By symmetry, this inequality is also necessary and sufficient for player 3's strategy to be optimal. Finally, $\frac{b}{c} - r \geq \frac{c}{b}\left(1+r/2\right)$ if and only if $r \leq \frac{b-\alpha^* c}{c}$. For any r in this range, setting $\alpha = \alpha^*$ satisfies both incentive constraints.

11.2 Proof of Theorem A

Let $Y_i = \prod_{j\in N_i} Z_{i,j}$. I show that $E_{PRI}^{PUBCT}\left(Y\right) \supseteq E_{PUB}$.

Let σ^{PUB} be a SE strategy profile in game Γ_{PUB}. Define a strategy profile σ^{PRI} in game $\Gamma_{PRI}^{PUBCT}\left(Y\right)$ as follows:

- $\sigma_i^{PRI}\left(h^{0-}\right) = \sigma_i^{PUB}\left(h^{0-}\right)$.

- $\sigma_i^{PRI}(h^{t+}) = (z_{i,j,t})_{j \in N_i}$ for all t.

- Let $\hat{z}_{i,j,t}$ be the $\{i,j\}$ coordinate of $y_{i,t}$. At history h_i^{t-}, if $\hat{z}_{i',j',\tau} = \hat{z}_{j',i',\tau}$ for all $\{i',j'\} \in L$ and all $\tau < t$, then let $\sigma_i^{PRI}(h_i^{t-}) = \sigma_i^{PUB}(\hat{h}_i^t)$, where $\hat{h}_i^t = \left(a_{i,\tau}, (\hat{z}_{i',j',\tau})_{\{i',j'\} \in L}\right)_{\tau=0}^{t-1}$. If instead $\hat{z}_{i',j',\tau} \neq \hat{z}_{j',i',\tau}$ for some $\{i',j'\} \in L$ and $\tau < t$, then let $\sigma_i^{PRI}(h_i^{t-}) = \alpha_i^*$.

Clearly, σ^{PRI} yields the same payoff vector as σ^{PUB}. It remains only to show that σ^{PRI} is a SE profile. To see this, note first that player i does not have a profitable deviation at an action history h_i^{t-} with $\hat{z}_{i',j',\tau} = \hat{z}_{j',i',\tau}$ for all $\{i',j'\} \in L$ and all $\tau < t$. This follows because her continuation payoff from playing any action a_i at history h_i^{t-} under σ^{PRI} is the same as her continuation payoff from playing a_i at history $\hat{h}_i^t$ under σ^{PUB}, and σ^{PUB} is a SE. Second, player i does not have a profitable deviation at an action history h_i^{t-} with $\hat{z}_{i',j',\tau} \neq \hat{z}_{j',i',\tau}$ for some $\{i,j\} \in L$ and $\tau < t$, because starting from such a history her opponents play α_{-i}^* forever and α_i^* is a best response to α_{-i}^*. Finally, player i does not have a profitable deviation at a communication history h^{t+}. In the case where $\hat{z}_{i,j,\tau} = \hat{z}_{j,i,\tau}$ for all $\{i,j\} \in L$ and all $\tau < t$, this follows because her continuation payoff when she conforms to σ^{PRI} equals her expected continuation payoff under σ^{PUB} conditional on reaching history $\hat{h}_i^t$, playing $a_{i,t}$, and observing signals $(z_{i,j,t})_{j \in N_i}$, while her continuation payoff when she deviates equals $u_i(\alpha^*)$, which is weakly less. In the case where $\hat{z}_{i,j,\tau} \neq \hat{z}_{j,i,\tau}$ for some $\{i,j\} \in L$ and $\tau < t$, this follows because her continuation payoff equals $u_i(\alpha^*)$ whether she conforms or deviates. Hence, player i does not have a profitable deviation at any history, so σ^{PRI} is a SE profile.

11.3 Proof of Theorem B

I first introduce one unorthodox piece of terminology. Throughout the appendix, I will say that an (action or communication) history h_i^t is *on-path* under strategy profile σ if it reached with positive probability when all players follow σ *or* if there exists another history $\tilde{h}_i^t$ that differs from h_i^t only in player i's past actions $(a_{i,\tau})_{\tau=0}^t$ such that $\tilde{h}_i^t$ is reached with positive probability when all players follow σ. A history is *off-path* otherwise. The point of this terminology is that if player i "trembles" at an action history under a SE σ but nonetheless an on-path signal is generated, then player i will want to "forget" about the deviation. By calling the resulting history "on-path," it will be possible to insist that player i plays her mutual-minmax action α_i^* at "off-path" histories, which is convenient for constructing equilibria.

Let $Y_{i,j} = \prod_{\{i',j'\} \in L} (Z_{i',j'} \cup \{1_{i',j'}\}) \cup \{0\}$, where 0 and $1_{i',j'}$ are arbitrary disjoint messages not contained in any $Z_{i',j'}$. If a message $y_{i,j} \neq 0$ and the $\{i',j'\}$ coordinate of $y_{i,j}$ is an element of $Z_{i',j'}$ (rather than $1_{i',j'}$), then I refer to it as an $\{i',j'\}$ *report*. I show that $E_{PRI}^{PRICT}(Y) \supseteq E_{PUB}$.

Let σ^{PUB} be a SE strategy profile in game Γ_{PUB}. I now construct a strategy profile σ^{PRI} in game $\Gamma_{PRI}^{PRICT}(Y)$ which will be shown to be a SE profile with the same payoff vector as σ^{PUB}. I first describe play at action histories, then describe play at on-path communication histories, and finally describe play at off-path communication histories.

Action Histories: $\sigma_i^{PRI}(h^{0-}) = \sigma_i^{PUB}(h^{0-})$. At subsequent on-path action histories, $\sigma_i^{PRI}(h_i^{t-}) = \sigma_i^{PUB}(\hat{h}_i^t)$, where $\hat{h}_i^t = \left(a_{i,\tau}, (\hat{z}_{i',j',\tau})_{\{i',j'\} \in L}\right)_{\tau=0}^{t-1}$ and $\hat{z}_{i',j',\tau}$ is the $\{i',j'\}$ report player i received in period τ. If player i received conflicting $\{i',j'\}$ reports in some period $\tau < t$, or did not receive an $\{i',j'\}$ report in some period $\tau < t$, then h_i^{t-} is an off-path history (as will become clear from the description of the communication phase below). At off-path action histories, $\sigma_i^{PRI}(h_i^{t-}) = \alpha_i^*$.

On-Path Communication Histories: In round 1, send message $\left((z_{i,j,t})_{j \in N_i}, (1_{i,j,t})_{j \notin N_i}\right)$ to every player $j' \in N_i$. In subsequent rounds, partition histories by whether they contain *new information* or *no new information*: a history $h_i^{t,k}$ has *new information* if $y_{j,i}^k \notin \left\{\{y_{j,i}^{k'}\}_{k'<k}, \emptyset_{j,i}\right\}$ for some $j \in N_i$, and has *no new information* if $y_{j,i}^k \in \left\{\{y_{j,i}^{k'}\}_{k'<k}, \emptyset_{j,i}\right\}$ for all $j \in N_i$. That is, a history has no new information for i if every player's most recent message to i was either a copy of one of his previous messages or was "no message," and has new information otherwise. At histories with no new information, player i sends $\emptyset_{i,j}$ to all $j \in N_i$. At histories with new information, player i sends every $j \in N_i$ the message with $\{i',j'\}$ report $\hat{z}_{i',j',t}$ if all $\{i',j'\}$ reports that she has sent or received in earlier rounds equal $\hat{z}_{i',j',t}$, and with $\{i',j'\}$ coordinate $1_{i',j',t}$ if she has not yet received an $\{i',j'\}$ report.

Off-Path Communication Histories: If history $h_i^{t,k}$ has new information, if player i deviated from σ^{PRI} in round $k-1$, or if $k=0$, send message 0 to all $j \in N_i$. Otherwise, send $\emptyset_{i,j}$ to all $j \in N_i$.

Note that if all players follows σ^{PRI}, then for every player $i \in N$ and any on-path history h_i^t, $\sigma_i^{PRI}\left(h_i^{t-}\right) = \sigma_i^{PUB}\left(\hat{h}_i^t\right)$, where $\hat{h}_i^t = \left(a_{i,\tau}, \left(z_{i',j',\tau}\right)_{\{i',j'\}\in L}\right)_{\tau=0}^{t-1}$, and in addition every communication phase ends in finitely many rounds. Therefore, σ^{PRI} yields the same payoff vector as σ^{PUB}. It remains to show that σ^{PRI} is a SE profile.[31]

I first claim that if any player i deviates from σ^{PRI} at any communication history $h_i^{t,k}$, then every player $j \neq i$ plays α_j^* in all subsequent periods.

The first step in proving the claim is showing that if player i deviates from σ^{PRI} at any communication history $h_i^{t,k}$, then some other player reaches an off-path history during the period t communication phase. This is clearly true if player i deviates by sending message 0, as message 0 is never sent on-path. It is also true if player i deviates by sending (to some $j \in N_i$) a message with $\{i', j'\}$ coordinate $1_{i',j',t}$ rather than sending an $\{i', j'\}$ report, sending an $\{i', j'\}$ report rather than $1_{i',j',t}$, or sending $\emptyset_{i,j}$ rather than a non-empty message, as player j knows at what rounds $h_i^{t,k}$ has new information on-path and at what rounds player i has received a $\{i', j'\}$ report on-path. The only remaining possibility is that player i deviates by sending an $\{i', j'\}$ report $\hat{z}_{i',j',\tau} \neq z_{i',j',\tau}$ to some $j \in N_i$. Assume without loss of generality that $j' \neq i$. Let $(j', j_1, \ldots, j_m, j)$ be a path from j' to j that does not include i, which exists by 2-connectedness. Then in round 1 player j' sends $\{i', j'\}$ report $z_{i',j',\tau}$ to player j_1, and by induction in round $m' + 1$ player $j_{m'}$ either sends $\{i', j'\}$ reports $z_{i',j',\tau}$ to player $j_{m'+1}$ or sends message 0 to player $j_{m'+1}$. In either case, player j receives either $\{i', j'\}$ report $z_{i',j',\tau}$ or message 0 in round $m + 1$, so $h_j^{t,\max\{k+1,m+1\}}$ is an off-path history.

The second—and final—step in proving the claim is showing that if all players except possibly i conform to σ^{PRI} and some player $j \neq i$ reaches an off-path history during the period t communication phase, then every player $j' \neq i$ plays $\alpha_{j'}^*$ in all subsequent periods. To see this, note that the first off-path history reached by a player j during the period t communication phase (call it $h_j^{t,k}$) has new information, so at $h_j^{t,k}$ player j sends message 0 to all of his neighbors in round $k+1$. Thus, if a player $j \neq i$ reaches an off-path history during the period t communication phase, then so do all of his neighbors. Since for all $j, j' \neq i$ there is a path from j to j' excluding i, it follows that every $j' \neq i$ reaches an off-path history

[31] As in the proof of Theorem A, I will show that $\left(\sigma^{PRI}, \mu\right)$ is a SE for any consistent belief system μ.

during the period t communication phase. Therefore, every subsequent action history is off-path for all $j' \neq i$, so all $j' \neq i$ play $\alpha^*_{j'}$ in all subsequent periods.

It follows from the claim that no player has a profitable deviation at an on-path history. For at any on-path action history h_i^{t-} player i's continuation payoff from playing any action a_i is the same as her continuation payoff from playing a_i at history $\hat{h}_i^t$ under σ^{PUB}. And at any on-path communication history $h_i^{t,k}$ player i's continuation payoff from conforming to σ^{PRI} equals her continuation payoff under σ^{PUB} conditional on reaching history $\hat{h}_i^t$, playing $a_{i,t}$, and observing some subset of the period t signals $(z_{i,j,t})_{\{i,j\} \in L}$, while her continuation payoff from deviating equals $u_i(\alpha^*)$, which is weakly less.

Finally, I argue that no player has a profitable deviation at an off-path history. The key observation is that if player i is at an off-path history then regardless of her future play all of her opponents will play α^* in every subsequent period. This is immediate from the claim if player i is the only player that has deviated from σ^{PRI} and player i has deviated at a communication history. If player i deviated from σ^{PRI} only at an action history and an off-path signal $z_{i,j}$ was generated, then player j is at an off-path history.[32] Similarly, if some player $j \neq i$ has deviated from σ^{PRI}, then that player is at an off-path history. In either of these cases, the second paragraph of the proof of the claim then implies that all players $j' \neq i$ will reach an off-path history by the end of the communication phase and will subsequently play α^* forever. Therefore, if i conforms to σ^{PRI} her continuation payoff is $u_i(\alpha^*)$ (noting that each communication phase ends in finite time starting from any vector of histories when all players subsequently conform to σ^{PRI}), while if she deviates her continuation payoff is weakly less.

11.4 Proof of Theorem 1

Let $Y_{i,j} = \prod_{\{i',j'\} \in L} (Z_{i',j'} \cup \{1_{i',j'}\}) \cup \{0\}$, as in the proof of Theorem B. To define m^0, first let L' be an arbitrary spanning tree of L (i.e., a connected subnetwork with no cycles that contains all the nodes in L), and let $N_i' \subseteq N_i$ be the set of player i's neighbors in L'. Renumber the players such that the leaves of L' (i.e., the players with only one neighbor in

[32] If player i deviated at an action history and an on-path signal was generated, then the history is classified as on-path.

L') are numbered $1, 2, \ldots, n'$; these players are subsequently referred to as *leaf players*. Now define m^0 by letting $m_i^0 = 4n' |Z|$ for all $i \in \{1, \ldots, n'\}$ and $m_i^0 = 0$ for all $i \in \{n'+1, \ldots, n\}$. In particular, only leaf players start with money. In addition, number the elements of Z from 1 to $|Z|$. I show that $E_{PRI}^{\$}(Y, m^0) \supseteq E_{PUB}$.

Let σ^{PUB} be a SE profile in Γ_{PUB}. I construct a profile σ^{PRI} in $\Gamma_{PRI}^{\$}(Y, m^0)$ which will be shown to be an SE profile with the same payoffs as σ^{PUB}. I first describe on-path play in the action phase and then describe on-path play in the communication phase (which is now broken into a "reporting subphase" followed by a "confirmation subphase"), off-path play, and off-path beliefs.

Actions: $\sigma_i^{PRI}(h^{0-}) = \sigma_i^{PUB}(h^{0-})$. In subsequent periods, $\sigma_i^{PRI}\left(h_i^{t-}\right) = \sigma_i^{PUB}\left(\hat{h}_i^t\right)$, where $\hat{h}_i^t = \left(a_{i,\tau}, (\hat{z}_{i',j',\tau})_{\{i',j'\} \in L}\right)_{\tau=0}^{t-1}$ and $\hat{z}_{i',j',\tau}$ is the $\{i',j'\}$ report player i received in period τ. If player i received conflicting $\{i',j'\}$ reports or did not receive an $\{i',j'\}$ report in some period $\tau < t$, then h_i^{t-} is an off-path history (as will become clear from the description of the communication phase below) and $\sigma_i^{PRI}\left(h_i^{t-}\right)$ is therefore given by the description of off-path play below.

Reporting Subphase: There are $n-1$ rounds of reporting in which players report all signals that they have been informed of and do not transfer money. In particular, in round 1 player i sends message $\left((z_{i,j,t})_{j \in N_i}, (1_{i,j,t})_{j \notin N_i}\right)$ to every player $j' \in N_i'$. In rounds 1 through $n-2$, player i sends message with $\{i',j'\}$ report $\hat{z}_{i',j',t}$ if all $\{i',j'\}$ reports she has sent or received in earlier rounds equal $\hat{z}_{i',j',t}$, and with $\{i',j'\}$ coordinate $1_{i',j',t}$ if she has not yet received an $\{i',j'\}$ report. Note that if players conform then all players are informed of the true signals in the reporting subphase.

Confirmation Subphase: I first describe the confirmation subphase at time $t = 0$, and then describe how it differs at time $t \geq 1$.

If player 1 has received consistent (i.e., non-conflicting) $\{i,j\}$ reports for all $\{i,j\} \in L$, and these reports equal $\left(\hat{z}_{i,j}^1\right)_{\{i,j\} \in L}$, then player 1 sends $\$k_1$ to the next player on the path from player 1 to player 2 in L' (note that this is well-defined because L' is a tree), where k is the number between 1 and $|Z|$ assigned to $(\hat{z}_{i,j})_{\{i,j\} \in L}$. In the next round, this player passes all money received from player 1 on to the next player on the path from player 1 to 2 in L'. This process continues until the money reaches player 2. When player 2 receives $\$k_2$ (where $k_2 = k_1$ on-path), then if player 2 has received consistent $\{i,j\}$ reports for all $\{i,j\} \in L$ and in addition k_2 is the number assigned to $\left(\hat{z}^2_{i,j}\right)_{\{i,j\}\in L}$, player 2 then sends $\$2k_2$ to the next player on the path from player 2 to player 3 in L'. Continue this process: when player $i \in \{2,\ldots,n'-1\}$ receives $\$k_i$, she sends $\$i\,(k_i/\,(i-1))$ down the path to player $i+1$ if $k_i/\,(i-1)$ is the number assigned to $\left(\hat{z}^i_{i',j'}\right)_{\{i',j'\}\in L}$. When player n' receives $\$k_{n'}$, she sends $\$n'\,(k_{n'}/\,(n'-1))$ down the path back to player 1 if $k_{n'}/\,(n'-1)$ is the number assigned to $\left(\hat{z}^{n-1}_{i,j}\right)_{\{i,j\}\in L}$.

Finally, if player 1 gets back $\$n'k_1$ (where $\$k_1$ is the amount of money she initially sent toward player 2), she then sends $\$2n'\,|Z|$ down the path to player 2. As above, this money is passed along the path from player 1 to player 2. If player 2 receives $\$2n'\,|Z|$, he sends the same amount back toward player 1. If player 1 receives $\$2n'\,|Z|$ back, she then sends $\$2n'\,|Z|$ down the path to player 3, and so on until she sends the $\$2n'\,|Z|$ toward player n' and player n' sends it back.

At times $t \geq 1$, the confirmation subphase is as above, except that throughout $\$k_i$ is replaced with $\$k_i/2^t$, and $\$2n'\,|Z|$ is replaced with $\$\,(4-1/2^{t-1})\,n'\,|Z|$.[33]

Off-Path Play: Define the $\{i,j\}$-*connectors* of network L, denoted $C_{i,j}$, to be the set of players that lie on a path from i to j in L (including i and j themselves). For players i and $j \in N_i$, say that player i *punishes* player j starting at history h^t_i if she plays $\alpha^*_{i,j'}$ in all subsequent periods for all players $j' \in N_i \cap C_{i,j}$, in the next communication round sends message 0 to all players $j' \in N_i \cap C_{i,j}$, and never again transfers money to any player $j' \in N_i \cap C_{i,j}$. I specify the following aspects of off-path play:

1. If player i ever receives an off-path signal $z_{i,j}$, $\{i',j'\}$ report $\hat{z}_{i',j'}$ or $1_{i',j'}$, or transfer $m_{j,i}$ from player j at an on-path history h^t_i (i.e., a signal, report, or transfer that is never received from j at h^t_i under σ^{PRI}), then player i punishes player j.

[33] Replacing $\$k_i$ with $\$k_i/2^t$ prevents players $2,\ldots,n'$ from running out of money, which ensures that the prescribed actions are feasible. Replacing $\$2n'\,|Z|$ with $\$2\,(2-1/2^t)\,n'\,|Z|$ ensures that the last confirmation phase transfers get larger every period. This is important for sequential rationality, as will become clear.

2. If player i ever sends an off-path $\{i', j'\}$ report $\hat{z}_{i',j'}$ or $1_{i',j'}$ or transfer $m_{i,j}$ to player j at an on-path history h_i^t (i.e., a report or transfer that is never sent to j at h_i^t under σ^{PRI}), then player i punishes player j.

3. If player i receives message 0 from player j or sends message 0 to player j at any history h_i^t (on or off-path), then player i punishes player j.

4. If player i receives a transfer $m_{j,i} > 0$ from a player $j \notin N_i'$ or sends a transfer $m_{i,j} > 0$ to a player $j \notin N_i'$ at any history h_i^t (on or off-path), then player i punishes player j.

5. If player i receives a transfer $m_{j,i} \notin \left\{0, \frac{1}{2^t}, \frac{2}{2^t}, \ldots, \frac{n'|Z|}{2^t}, \left(4 - \frac{1}{2^{t-1}}\right) n' |Z|\right\}$ from player j or sends a transfer $m_{i,j} \notin \left\{0, \frac{1}{2^t}, \frac{2}{2^t}, \ldots, \frac{n'|Z|}{2^t}, \left(4 - \frac{1}{2^{t-1}}\right) n' |Z|\right\}$ to player j at any period t history h_i^t (on or off-path), then player i punishes player j.

6. Player i never sends a transfer $m_{i,j} > 0$ to a player $j \notin N_i'$. That is, players never transfer money to players to whom they are not linked in L', even off-path.

7. Player i never sends a transfer $m_{i,j} \notin \left\{0, \frac{1}{2^t}, \frac{2}{2^t}, \ldots, \frac{n'|Z|}{2^t}, \left(4 - \frac{1}{2^{t-1}}\right) n' |Z|\right\}$ to any player j at any period t history. That is, at off-path period t histories the players only make transfers that they make at some on-path period t history.

All other aspects of off-path play are defined implicitly as follows. Consider the auxiliary game in which player i receives payoff $u_i(\alpha^*) - 1$ if she deviates from the on-path play or the aspects of off-path play specified above. This game has a sequential equilibrium strategy profile by standard arguments.[34] Let the other aspects of off-path play be given by any such strategy profile.

Off-Path Beliefs: I specify that whenever player i receives an off-path report, transfer, or message from player j at an on-path communication history, she believes that player

[34] A subtlety here is that players' action sets in the communication phase are countably infinite, as $m_{i,j}$ can take on any rational value less than m_i. However, the specification of off-path play implies that any period t transfer $m_{i,j} \notin \left\{0, \frac{1}{2^t}, \frac{2}{2^t}, \ldots, \frac{n'|Z|}{2^t}, \left(4 - \frac{1}{2^{t-1}}\right) n' |Z|\right\}$ yields continuation value $u_{i,j'}\left(\alpha^*_{i,j'}, \alpha^*_{j',i}\right)$ for all $j' \in C_{(i,j)}$. In addition, player i's continuation value against players $j' \notin C_{(i,j)}$ is non-decreasing in player i's money holding, as a player with more money can mimic any strategy that is feasible for a player with less money. So any period t transfer $m_{i,j} \notin \left\{0, \frac{1}{2^t}, \frac{2}{2^t}, \ldots, \frac{n'|Z|}{2^t}, \left(4 - \frac{1}{2^{t-1}}\right) n' |Z|\right\}$ is "weakly dominated" by $m_{i,j} = 0$ (in that it never yields a strictly higher continuation value than $m_{i,j} = 0$). Therefore, players' action sets in the auxilliary game have only finitely many undominated actions, and such games admit sequential equilibria (by applying Kakutani's fixed point theorem to the game with only undominated actions).

j just deviated, and moreover that player j also just sent message 0 to all players $j' \in N_j \backslash \{i\}$. Consistent belief systems with this property exist, since players may believe that deviations in later rounds are infinitely more likely than deviations in earlier rounds, and that deviations where player j sends message 0 to all $|N_j|$ of his neighbors are infinitely more likely than deviations where he sends message 0 to $|N_j|-1$ of his neighbors, and that these deviations are in turn infinitely more likely than deviations where he sends message 0 to fewer than $|N_j|-1$ of his neighbors.

It is clear that σ^{PRI} yields the same payoffs as σ^{PUB}. I will show that σ^{PRI} together with any consistent belief system with the specified property is a SE.

A preliminary observation, which I will use repeatedly, is that leaf players never send transfers at off-path histories. To see this, fix a leaf player i. At any off-path history h_i^t, player i is punishing some player j (as i either sent or received an off-path signal, transfer, report, or message to or from some j). Since L' is connected and spans L, $N_i \subseteq C_{i,j}$, so i never again transfers money to any player in N_i.[35]

I now prove a key lemma, which says that if player i deviates from σ^{PRI} then each of her neighbors either minmaxes her or plays as if she had conformed to σ^{PRI}.

Lemma 1 *For every pair of players i and $j \in N_i$, every strategy σ_i, and every action history h_j^{t+1-} reached under strategy profile $(\sigma_i, \sigma_{-i}^{PRI})$, $\sigma_{j,i}^{PRI}(h_j^{t+1-}) \in \left\{\alpha_{j,i}^*, \sigma_{j,i}^{PUB}\left(\hat{h}_j^{t+1}\right)\right\}$, where $\hat{h}_j^{t+1} = \left(a_{j,\tau}, (z_{i',j',\tau})_{\{i',j'\}\in L}\right)_{\tau=0}^{t}$.*

Proof. Suppose, toward a contradiction, that $\sigma_{j,i}^{PRI}(h_j^{t+1-}) \notin \left\{\alpha_{j,i}^*, \sigma_{j,i}^{PUB}\left(\hat{h}_j^{t+1}\right)\right\}$ for some $j \in N_i$. Note that if j ever received an off-path signal, transfer, report, or message, then the player j' who he received it from must be in $C_{i,j}$ (since only i deviates from σ^{PRI}), so j plays $\alpha_{j,i}^*$ (as j minmaxes every player in $C_{j,j'}$, and since $j \in N_i$ there is a path of the form $(j, i, \ldots, j')$). Hence, history h_j^{t+1-} must be on-path, and there must be a period $t' \leq t$ such that in the period t'communication phase j received a consistent vector of $\{i', j'\}$ reports that does not equal $(z_{i',j',t'})_{\{i',j'\}\in L}$. I consider three cases, deriving a contradiction in each:

Case 1: Player i is a leaf player. For player j to be at an on-path history with incorrect reports, it must be that player i sent an off-path signal, transfer, report, or message to

[35] If i is a non-leaf player, then $N_i \backslash C_{i,j} \neq \emptyset$ is possible.

some player $j' \in N_i \backslash \{j\}$ in some period $t' \leq t$ (note that it is not possible that i sent an incorrect but on-path report to j', because the fact that j is i's only neighbor in L' implies that all reports sent from i to j' are off-path). Since i is a leaf player, $j \in C_{i,j'}$. So player j receives message 0 from some player in $C_{i,j'}$ during the period t' communication phase. This contradicts the fact that j has received no off-path messages.

Case 2: Player i is not a leaf player, and history $h_1^{t',0}$ is off-path. Let $t_0 \leq t'$ be the first time τ such that history $h_1^{\tau,0}$ is off-path. I will show that player j does not receive the $\$\,(4 - 1/2^{t_0-1})\, n' \, |Z|$ transfer in period t_0, which contradicts hypothesis that history h_j^{t+1-} is on-path.

First note that if any non-leaf player has any money at the beginning of period $t_0 - 1$, then $h_1^{t_0-1,0}$ is off-path, because if any non-leaf player did not pass on some of any of the transfers he received in some period $\tau \leq t_0 - 2$ then player 1 did not get back either the $\$ n' k_1 / 2^\tau$ transfer or the $\$\,(4 - 1/2^{\tau-1})\, n' \, |Z|$ transfer in period τ (recalling that all leaf players are following σ^{PRI}). So by definition of t_0 no non-leaf has any money at the beginning of period $t_0 - 1$.

I now claim that the joint money holdings of all non-leaf players at the beginning of period t_0 is at most $\$\,(4 - 1/2^{t_0-2})\, n' \, |Z|$ (i.e., $\sum_{j=n'+1}^{n} m_j \leq (4 - 1/2^{t_0-2})\, n' \, |Z|$). To see this, suppose that the non-leaf players collectively try to maximize their joint money holdings in the period $t_0 - 1$ communication phase. Note that every dollar that the non-leaf players do not pass on to a leaf player out of any on-path transfer they receive reduces the size of the next on-path transfer sent by a leaf player by more than one dollar, and that leaf players do not send transfers at off-path histories. So the joint money holdings of the non-leaf players is maximized when they pass on all on-path transfers except the last one, which is of size $\$\,(4 - 1/2^{t_0-2})\, n' \, |Z|$.

Now if player j receives a transfer of size $\$\,(4 - 1/2^{t_0-1})\, n' \, |Z|$ in period t_0 it must be that the joint money holdings of the non-leaf players (including player j if he is a non-leaf player) reaches $\$\,(4 - 1/2^{t_0-1})\, n' \, |Z|$ at some point during period t_0. However, it can be seen that the joint money holdings of the non-leaf players at any point in period t_0 is no more than $\$\,(4 - 1/2^{t_0-2})\, n' \, |Z| + (n' - 1)\, |Z| \, / 2^{t_0}$, since they start the period with at most $\$\,(4 - 1/2^{t_0-2})\, n' \, |Z|$ and can obtain at most $\$\,(n' - 1)\, |Z| \, / 2^{t_0}$ more in the course of the communication phase (by sending $\$\left|Z\right|/2^{t_0}$ to player 2 in the appropriate round and eventually receiving $\$n'\left|Z\right|/2^{t_0}$ from player n'). Finally,

$$\begin{aligned}
& \left(4-1/2^{t_0-2}\right)n'\left|Z\right|+\left(n'-1\right)\left|Z\right|/2^{t_0} \\
= & \ 4n'\left|Z\right|-\left(1/2^{t_0}\right)\left(3n'+1\right)\left|Z\right| \\
< & \ 4n'\left|Z\right|-\left(1/2^{t_0}\right)2n'\left|Z\right| \\
= & \left(4-1/2^{t_0-1}\right)n'\left|Z\right|.
\end{aligned}$$

So player j does not receive the $\$\left(4-1/2^{t_0-1}\right)n'\left|Z\right|$ transfer in period t_0.

Case 3: Player i is not a leaf player, and history $h_1^{t',0}$ is on-path. If player 1 does not receive a consistent vector of reports in the period t' reporting subphase, then the argument is as in Case 2. So suppose that she does, and denote this vector by $\left(\hat{z}_{i',j'}\right)_{\{i',j'\}\in L}$. Note that it is not possible for all players other than i to have the same consistent—but incorrect—vector of reports at the start of the period t' confirmation phase (as if $\hat{z}_{i',j'}\neq z_{i',j'}$ then players i' and j' cannot have consistent vector $\left(\hat{z}_{i',j'}\right)_{\{i',j'\}\in L}$). So there is some player i' who at the start of confirmation phase is either off-path or is on-path with consistent vector $\left(\tilde{z}_{i',j'}\right)_{\{i',j'\}\in L}\neq\left(\hat{z}_{i',j'}\right)_{\{i',j'\}\in L}$. Let k be number associated to $\left(\hat{z}_{i',j'}\right)_{\{i',j'\}\in L}$, let $k'\neq k$ number associated to $\left(\tilde{z}_{i',j'}\right)_{\{i',j'\}\in L}$. Consider two cases:

1. $k<k'$: I claim that the first transfer player i' receives in the period t' confirmation phase is off-path. To see this, note that no non-leaf player begins period t' with money (because $h_1^{t',0}$ is on-path), so for no joint strategy of the non-leaf players is their joint money holding at the round where player i' receives her first on-path transfer greater than $k/2^{t'}$ times the number of leaf players who send transfers prior to this round. Since i' expects a transfer of $k'/2^{t'}$ times this number (unless she is off-path already), the first transfer she receives is off-path. Consequently, player i' punishes the player j' who sends the off-path transfer, and in particular never again transfers money to him or to any other player in $C_{i',j'}$. Since there are no transfers along links not in L' even off-path, this implies that the player 1 does not receive her expected $\$n'k/2^{t'}$ transfer, and therefore player 1 does not send the $\$\left(4-1/2^{t'-1}\right)n'\left|Z\right|$ transfer. Finally, as argued in Case 2, the non-leaf players can collectively obtain no more than

$\$\,(n'-1)\,|Z|\,/2^{t'} < \$\left(4-1/2^{t'-1}\right)n'\,|Z|$ in the course of the period t' confirmation phase, so it follows that player j does not receive the $\$\left(4-1/2^{t'-1}\right)n'\,|Z|$ transfer, a contradiction.

2. $k > k'$: If the first transfer player i' receives in the period t' confirmation phase is off-path, the argument is as in Case 2. If it is on-path, then it equals $k'/2^{t'}$ times the number of leaf players who send transfers prior to this round. Now let l be a leaf player such that i' is on unique path from l to 1 in L', with $l = i'$ if i' is a leaf player herself. Then the first transfer l receives in the period t' confirmation phase is at most $k'/2^{t'}$ times the number of leaf players who send transfers prior to this round. So l then sends a transfer that is at most $k'/2^{t'}$ greater than the transfer she received in the preceding round. I now claim that player 1 does not receive her expected $\$n'k/2^{t'}$ transfer in the period t' confirmation phase. For the non-leaf players begin period t' with no money, no leaf player sends a transfer that is more than $k/2^{t'}$ greater than what she received, and some leaf player (player l) sends a transfer that is only at most $k'/2^{t'}$ greater than what she received. The rest of the argument is as in Case 2.

■

Lemma 1 is not quite enough to rule out on-path deviations. The following lemma will also be needed.

Lemma 2 *Suppose that under strategy profile $\left(\sigma_i, \sigma_{-i}^{PRI}\right)$ an off-path action history h_j^{t-} is reached for some $j \in N_i$. Then $\sigma_{j',i}^{PRI}\left(h_{j'}^{t'-}\right) = \alpha_{j',i}^*$ for all $t' > t$ and all $j' \in N_i$.*

Proof. I first note that it suffices to show that if h_j^{t-} is off-path for some $j \in N_i$ then the next action history $h_{j'}^{t+1-}$ is off-path for all $j' \in N_i$. For if $h_{j'}^{t+1-}$ is off-path then the first off-path signal, transfer, report, or message received by player j' must have come from a player $j'' \in C_{i,j'}$, and since $j' \in N_i$ it follows that $i \in C_{j',j''}$.

I now show that $h_{j'}^{t+1-}$ is off-path for all $j' \in N_i$, considering three cases.

Case 1: Player i is a leaf player. Since h_j^{t-} is off-path, player i sent an off-path signal, transfer, report, or message to some player $j' \in N_i$ at some time $t' < t$. Since i is a leaf player, $N_i \subseteq C_{i,j'}$, so every player $j'' \in N_i$ receives message 0 in the period t' communication phase. So $h_{j''}^{t+1-}$ is off-path for all $j'' \in N_i$.

Case 2: Player i is not a leaf player and history $h_1^{t,0}$ is off-path. The same argument as in Case 2 of the proof of Lemma 1 implies that no player $j' \in N_i$ receives the $\$\,(4 - 1/2^{t_0 - 1})\, n' \, |Z|$ transfer in the first period t_0 at which $h_1^{t_0,0}$ is off-path, so $h_{j'}^{t+1-}$ is off-path for all $j' \in N_i$.

Case 3: Player i is not a leaf player and history $h_1^{t,0}$ is on-path. Since i is the lone deviator and an off-path history h_j^{t-} is reached, i must have sent an off-path signal, message, report, or transfer to some player $j' \in N_i$ at some time $t' < t$. Note that $1 \notin C_{i,j'}$, as otherwise 1 would have received message 0 in the period t' communication phase and hence $h_1^{t,0}$ would be off-path. Now since L' is a spanning tree of L (so that in particular 1 does not lie on a path from i to j' in L') there is a leaf player l such that the unique path from 1 to l in L' includes i and j' (where it may be that $j' = l$). Now no transfer from l reaches 1 in the period t communication phase, as if i comes before j' in the path from 1 to l then any transfer sent by l does not reach 1, and if j' comes before i then no transfer reaches l (as no transfers are made outside of L' and j' does not transfer money to any player in $C_{i,j'}$). So, recalling that no non-leaf player has money at the start of period t and that no leaf players deviate, it follows that player 1 does not receive her expected $\$n'k/2^t$ transfer in period t. Hence, no one receives the $\$\,(4 - 1/2^{t-1})\, n' \, |Z|$ transfer in period t, and therefore $h_{j''}^{t+1-}$ is off-path for all $j'' \in N$, and hence for all $j'' \in N_i$. ■

Together, Lemmas 1 and 2 imply that there are no profitable deviations at on-path histories: It is clear that there are no profitable deviations at on-path action histories, as playing any action a_i at an on-path action history h_i^{t-} under σ^{PRI} yields the same continuation payoff as does playing action a_i at history $\hat{h}_i^t$ under σ^{PUB}. Now suppose, toward a contradiction, that a player i has a profitable deviation at an on-path period t communication history. By Lemmas 1 and 2, such a deviation must lead some of i's neighbors to start minmaxing i in period $t+1$ and lead the rest of them to play $\sigma_{j,i}^{PUB}\left(\hat{h}_j^{t+1}\right)$ in period $t+1$ and then start minmaxing i in period $t+2$. Now such a deviation is weakly worse for i than conforming to σ^{PRI} in the period t communication phase, deviating to her myopic best response in the period $t+1$ action phase, and playing α_i^* from period $t+2$ on, since the latter deviation yields a weakly higher payoff in period $t+1$ (as best-responding to an arbitrary mixed action gives a weakly higher payoff than best-responding to the minmax mixed action) and the same payoff in all subsequent periods. But the latter deviation is not profitable, since there are no profitable deviations at on-path action histories, so the proposed deviation cannot be profitable, either.

Finally, I argue that there are no profitable deviations at off-path histories. Note that the only path in L from a player in $N_i \cap C_{i,j}$ to a player in $N_i \backslash C_{i,j}$ is the one through i, so if a (i,j)-game strategy $\sigma_{i,j}$ maximizes player i's payoff in the (i,j)-game for all $j' \in N_i \cap C_{i,j}$ and does not reduce the money transfer she receives from any player $j' \in N_i \cap C_{i,j}$ at any history then it maximizes her payoff overall (for fixed $\sigma_{i,-j}$). Now at an off-path action-phase history h_i^t where $\sigma_{i,j}$ is specified, player i plays $\alpha^*_{i,j}$ and believes that every player $j' \in N_i \cap C_{i,j}$ has received message 0 from some player in $C_{i,j'}$ (regardless of player i's strategy), so she expects every player $j' \in N_i \cap C_{i,j}$ to play $\alpha^*_{j',i}$ and make no transfers to her forever—hence, $\alpha^*_{i,j}$ is optimal. At an off-path communication-phase history where $\sigma_{i,j}$ is specified, player i sends message 0 and does not transfer money to player j and believes that every player $j' \in N_i \cap C_{i,j}$ will receive message 0 from some player $j' \in C_{i,j'}$ during the communication phase regardless of player i's strategy, by definition of $C_{i,j'}$ and the specification of off-path beliefs, so sending message 0 and not transferring money to player j is optimal. Finally, transferring $m_{i,j} > 0$ to a player $j \notin N'_i$ or transferring $m_{i,j} \notin \left\{0, \frac{1}{2^t}, \frac{2}{2^t}, \ldots, \frac{n'|Z|}{2^t}, \left(4 - \frac{1}{2^{t-1}}\right) n' |Z|\right\}$ to any player j in period t leads all $j' \in C_{i,j}$ to play $\alpha^*_{j',i}$ and make no transfers to i forever, so it is optimal for player i to never make such a transfer.

11.5 Proof of Proposition 1 (Sketch)

Let L' be such a spanning tree, and renumber the leaf players in L' by $1, \ldots, n'$, as in the proof of Theorem 1. Let $\varepsilon = \min_{i \in \{1,\ldots,n'\}} m_i^0$. Add a new "redistribution subphase" to the start of the period 0 communication phase. In it, all non-leaf players first pass all their money to player 1. Let $x = \sum_{i=n'+1}^{n} m_i^0$ be the joint initial money holding of the non-leaf players, so that player 1 receives \$$x$. The rest of the strategy profile is as in the proof of Theorem 1, except that throughout \$$k_i/2^t$ is replaced with \$$\frac{k_i}{2^t}\left(\frac{\varepsilon}{4n'|Z|}\right)$ and \$$\left(4 - 1/2^{t-1}\right) n' |Z|$ is replaced with \$$x + \left(1 - 1/2^{t+1}\right)\varepsilon$, reflecting the fact that players $2, \ldots, n'$ now end the redistribution subphase with as little as \$$\varepsilon$ rather than \$$4n'|Z|$ and player 1 ends the redistribution subphase with as little as \$$x + \varepsilon$ rather than \$$4n'|Z|$.

The proof that this is a SE profile is as in the proof of Theorem 1. In particular, the facts that non-leaf players end the redistribution subphase with no money and that the "confirmation transfer" $\$x + (1 - 1/2^{t+1})\,\varepsilon$ is greater than $\$x$ and increases each period imply that no player can mislead another about the history of signals.

11.6 Proof of Theorem 2

I first prove the result for "essential," and then describe how it must be modified for "strongly essential."

I start by introducing the notion of an M*-local public equilibrium* (M-LPE), where M is an arbitrary subnetwork of L. This is defined to be a SE in Γ_{PRI} in which $\sigma_{i,j}(h_i^t)$ depends only on $(z_{i,j,\tau})_{\tau=0}^{t-1}$ for all $\{i,j\} \in M$, and $\sigma_{i,j}(h_i^t)$ depends only on $\left((a_{i,j',\tau}, z_{i,j',\tau})_{\{i,j\} \in L \setminus M}\right)_{\tau=0}^{t-1}$ for all $i \in M$ and $j \notin M$. That is, a M-LPE is a SE in which players in M condition their play in a relationship with another player in M only on past play in that relationship, and condition their play in a relationship with a player outside M only on past play with players outside M. Denote the M-LPE payoff set in Γ_{PRI} by E_{PRI}^{MLPE}.

For the rest of the proof, assume that M is a nice subnetwork of L.

First, I claim that $E_{PRI} = E_{PRI}^{MLPE}$. The argument adapts the proof of Theorem 5.2 of Fudenberg and Levine (1994), which shows that the SE payoff set and PPE payoff set coincide in repeated games with imperfect public monitoring and a product structure. In particular, fix a SE σ in Γ_{PRI}, any let $\{i,j\} \in M$. Because M is a subtree of L, player i's beliefs at history h_i^t about player j's private history depend only on $(a_{i,j,\tau}, z_{i,j,\tau})_{\tau=0}^{t-1}$; this follows from consistency, because it holds for all completely mixed strategy profiles (by Bayes rule) and equalities are preserved in the limit. Given this, the fact that Γ_{PRI} has a product structure implies that player i's beliefs about player j's private history depend only on $(z_{i,j,\tau})_{\tau=0}^{t-1}$ (again by consistency, since for any completely mixed strategy profile player i's posterior over $(a_{j,i,\tau})_{\tau=0}^{t-1}$ given $(a_{i,j,\tau}, z_{i,j,\tau})_{\tau=0}^{t-1}$ depends only on $(z_{i,j,\tau})_{\tau=0}^{t-1}$, by Bayes rule). Now replace $\sigma_{i,j}$ with a strategy that depends only on $(z_{i,j,\tau})_{\tau=0}^{t-1}$ but has the same marginals over $A_{i,j}$ conditional on $(z_{i,j,\tau})_{\tau=0}^{t-1}$ as does $\sigma_{i,j}$. Do this for every $\{i,j\} \in M$. In addition, again because M is a subtree of L, for any $\{i,j\} \in L$ with $i \in M$ and $j \notin M$, player i's beliefs at history h_i^t about player j's private history depends only on $\left(\left(a_{i,j',\tau}, z_{i,j',\tau}\right)_{\{i,j\}\in L\backslash M}\right)_{\tau=0}^{t-1}$. For any such i, j, replace $\sigma_{i,j}$ with a strategy that depends only on $\left(\left(a_{i,j',\tau}, z_{i,j',\tau}\right)_{\{i,j\}\in L\backslash M}\right)_{\tau=0}^{t-1}$ but has the same marginals over $A_{i,j}$ conditional on $\left(\left(a_{i,j',\tau}, z_{i,j',\tau}\right)_{\{i,j\}\in L\backslash M}\right)_{\tau=0}^{t-1}$ as does $\sigma_{i,j}$. Then the resulting strategy profile (after both kinds of replacements) is a M-LPE with the same payoffs as σ; this is because for every pure strategy of any player i, she faces the same distribution over outcomes whether her opponents follows the original strategy profile or the modified strategy profile.

Second, I claim that $E_{PRI}^{MLPE} = E_{PRI}^{LPE}|_M + E_{PRI}|_{L\backslash M}$.[36] To see this, given a LPE σ' in $\Gamma_{PRI}|_M$ and a SE σ'' in $\Gamma_{PRI}|_{L\backslash M}$, define a strategy profile σ in Γ_{PRI} by letting $\sigma_{i,j}\left(h_i^t\right) = \sigma'_{i,j}\left(\left(z_{i,j,\tau}\right)_{\tau=0}^{t-1}\right)$ if $\{i,j\} \in M$ and $\sigma_{i,j} = \sigma''_{i,j}\left(\left(\left(a_{i,j',\tau}, z_{i,j',\tau}\right)_{\{i,j\}\in L\backslash M}\right)_{\tau=0}^{t-1}\right)$ if $\{i,j\} \in L\backslash M$. Then it is straightforward to check that σ is a M-LPE in Γ_{PRI}, so $E_{PRI}^{MLPE} \supseteq E_{PRI}^{LPE}|_M + E_{PRI}|_{L\backslash M}$. Similarly, given a M-LPE in Γ_{PRI}, σ, define strategy profiles σ' in $\Gamma_{PRI}|_M$ and σ'' in $\Gamma_{PRI}|_{L\backslash M}$ by $\sigma'_{i,j}\left(h_i^t\right) = \sigma_{i,j}\left(h_i^t\right)$ for all $\{i,j\} \in M$ and $\sigma''_{i,j}\left(h_i^t\right) = \sigma_{i,j}\left(h_i^t\right)$ for all $\{i,j\} \notin M$. Then σ' is a LPE in $\Gamma_{PRI}|_M$ and σ'' is a SE in $\Gamma_{PRI}|_{L\backslash M}$, so $E_{PRI}^{MLPE} \subseteq E_{PRI}^{LPE}|_M + E_{PRI}|_{L\backslash M}$. Combining the inclusions yields $E_{PRI}^{MLPE} = E_{PRI}^{LPE}|_M + E_{PRI}|_{L\backslash M}$.

Third, $E_{PRI}^{\$}|_M \supseteq E_{PUB}|_M$ by Theorem 1. In addition, $E_{PUB}|_M \backslash \operatorname{co}\left(E_{PRI}^{LPE}|_M\right) \neq \emptyset$ because M is nice, so $E_{PRI}^{\$}|_M \backslash \operatorname{co}\left(E_{PRI}^{LPE}|_M\right) \neq \emptyset$. In addition, $E_{PRI}^{\$}|_M \supseteq E_{PRI}|_M$ by the observation preceding Corollary 1, so $E_{PRI}^{\$}|_M \supseteq E_{PRI}^{LPE}|_M$.

Finally, given any message set and vector of initial money holdings $\left(\tilde{Y}, \tilde{m}^0\right)$ in $\Gamma_{PRI}|_M$, define message set and initial money holdings $\left(Y, m^0\right)$ in Γ_{PRI} by $Y_{i,j} = \tilde{Y}_{i,j}$ if $\{i,j\} \in M$, $Y_{i,j} = \emptyset$ if $\{i,j\} \notin M$, $m_i^0 = \tilde{m}_i^0$ if $i \in M$, and $m_i^0 = 0$ if $i \notin M$. Then $E_{PRI}^{\$}\left(Y, m^0\right) \supseteq E_{PRI}^{\$}\left(\tilde{Y}, \tilde{m}^0\right)|_M + E_{PRI}|_{L\backslash M}$, as given a SE σ' in $E_{PRI}^{\$}\left(\tilde{Y}, \tilde{m}^0\right)|_M$ and a SE σ'' in $E_{PRI}|_{L\backslash M}$ one can construct a SE σ in $E_{PRI}^{\$}\left(Y, m^0\right)$ by letting $\sigma_{i,j}\left(h_i^t\right) = \sigma'_{i,j}\left(\tilde{h}_i^t\right)$ if $\{i,j\} \in M$, where $\tilde{h}_i^t$ is derived from h_i^t by deleting actions, signals, messages, and transfers along links $\{i,j\} \notin M$, and letting $\sigma_{i,j}\left(h_i^t\right) = \sigma''_{i,j}\left(\hat{h}_i^t\right)$ if $\{i,j\} \notin M$, where $\hat{h}_i^t$ is derived from h_i^t by deleting actions and signals along links $\{i,j\} \in M$ and deleting all messages and transfers. Hence, $E_{PRI}^{\$} \supseteq E_{PRI}^{\$}|_M + E_{PRI}|_{L\backslash M}$.

[36] The notation here is that for sets $A, B \subseteq \mathbb{R}^n$, $A + B = \{a + b : a \in A, b \in B\}$.

Combining all the inclusions derived in the course of the proof, we have

$$E^{\$}_{PRI} \supseteq E^{\$}_{PRI}|_M + E_{PRI}|_{L\setminus M} \supsetneq E^{LPE}_{PRI}|_M + E_{PRI}|_{L\setminus M} = E^{MLPE}_{PRI} = E_{PRI},$$

where the strict inclusion uses the fact that, for any sets X, X', and W, if $X \supseteq X'$ and $X \setminus \mathrm{co}(X') \neq \emptyset$ then $X + W \supsetneq X' + W$.[37] Therefore, $E^{\$}_{PRI} \supsetneq E_{PRI}$.

The proof for "strongly essential" is almost identical. In place of a M-local public equilibrium, define a M-local cheap talk equilibrium to be a SE in Γ^{PRICT}_{PRI} in which players in M condition their play (including messages) in a relationship with another player in M only on past play in that relationship, and condition their play in a relationship with a player outside M only on past play with players outside M. Let E^{MLCTE}_{PRI} be the set of M-local cheap talk equilibrium payoffs in Γ^{PRICT}_{PRI}. Then $E^{PRICT}_{PRI} = E^{MLCTE}_{PRI}$ by the same argument as for $E_{PRI} = E^{MLPE}_{PRI}$, with the addition that strategies about which message to send may also need to be replaced by M-local cheap talk strategies with the same marginals. Next, $E^{MLCTE}_{PRI} = E^{LCTE}_{PRI}|_M + E_{PRI}|_{L\setminus M}$ by the same argument as for $E^{MLPE}_{PRI} = E^{LPE}_{PRI}|_M + E_{PRI}|_{L\setminus M}$, and $E^{\$}_{PRI}|_M \setminus \mathrm{co}\left(E^{LCTE}_{PRI}|_M\right) \neq \emptyset$ and $E^{\$}_{PRI}|_M \supseteq E^{LCTE}_{PRI}|_M$ by the same argument as for $E^{\$}_{PRI}|_M \setminus \mathrm{co}\left(E^{LPE}_{PRI}|_M\right) \neq \emptyset$ and $E^{\$}_{PRI}|_M \supseteq E^{LPE}_{PRI}|_M$ (where the statement that $E_{PUB}|_M \setminus \mathrm{co}\left(E^{LPE}_{PRI}|_M\right) \neq \emptyset$ is strengthened to $E_{PUB}|_M \setminus \mathrm{co}\left(E^{LCTE}_{PRI}|_M\right) \neq \emptyset$, which is possible when M is truly nice). Combining these inclusions with $E^{\$}_{PRI} \supseteq E^{\$}_{PRI}|_M + E_{PRI}|_{L\setminus M}$ as in the "essential" case yields $E^{\$}_{PRI} \supsetneq E^{PRICT}_{PRI}$.

11.7 Proof of Theorem 3

It is straightforward to verify that any network containing a subtree of size at three also contains a subtree of size exactly three. I claim that if $A_{i,j}$ is ε-dense for small enough ε then any subtree of size exactly three is truly nice.[38] The result then follows from Theorem 2.

To prove the claim, fix a subtree M of size three, which can always be written as $M = \{(1,2),(2,3)\}$ for $1,2,3 \in N$. Note that perfect monitoring games have a product structure,

[37] This follows from a separating hyperplane argument.

[38] In fact, if $A_{i,j}$ is ε-dense for small enough ε then any subtree of size at least three is truly nice. But it is slightly simpler to focus on subtrees of size exactly three.

so to show that M is truly nice I must only show that $E_{PUB}|_M \setminus \text{co}\left(E_{PRI}^{LCTE}|_M\right) \neq \emptyset$. I will do this by showing that player 1's greatest SE payoff in $\Gamma_{PUB}|_M$ is greater than her greatest LCTE payoff in $\Gamma_{PRI}|_M$.

I first derive an upper bound on player 1's greatest LCTE payoff in $\Gamma_{PRI}|_M$, which I denote by $\bar{u}_1^{PRI}$; that is, $\bar{u}_1^{PRI} = \sup u_{1,2}$, where the supremum is taken over LCTE in $\Gamma_{PRI}|_M$. I claim that $\bar{u}_1^{PRI}$ is bounded from above by the solution to the program

$$\max_{a_1,a_2} \frac{1}{2} f(a_1, a_2) - a_1$$

$$s.t. \;\; \frac{1}{2}\left(f(a_1, a_2) - (1-\delta) f(a_1, 0)\right) - a_2 \geq 0,$$

which is well-defined under our assumptions on f. Call this Program 1. To see this, let $\bar{a}_i$ be player i's expected discounted future action toward player $j \neq i$, for $i, j \in \{1, 2\}$:

$$\bar{a}_i \equiv (1-\delta) \sum_{s=0}^{\infty} \delta^s E\left[a_{i,t+s} \middle| \left(a_{i,\tau}, a_{j,\tau}\right)_{\tau=0}^{t-1}\right].$$ [39]

By concavity of f and Jensen's inequality, player i's continuation payoff in the (i, j)-game from conforming is at most $\frac{1}{2} f(\bar{a}_i, \bar{a}_j) - \bar{a}_i$. In addition, at some history player i's continuation payoff in the (i, j)-game from deviating to playing $a_i = 0$ forever must be at least $(1-\delta) \frac{1}{2} f(0, \bar{a}_j)$.[40] Hence, in any LCTE it follows that

$$(1-\delta) \frac{1}{2} f(0, \bar{a}_j) \leq \frac{1}{2} f(\bar{a}_i, \bar{a}_j) - \bar{a}_i,$$

or

$$\frac{1}{2}\left(f(\bar{a}_i, \bar{a}_j) - (1-\delta) f(0, \bar{a}_j)\right) - \bar{a}_i \geq 0.$$

Consequently, $\bar{u}_1^{PRI}$ is bounded from above by the solution to Program 1.

Next, I show that player 1's greatest SE payoff in $\Gamma_{PUB}|_M$ is greater than the solution to Program 1 if $A_{i,j}$ is ε-dense for small enough ε. I consider "grim-trigger" strategies profiles

[39] Note that this expectation is conditioned on information that is public to i and j.

[40] Technically, it must be at least $(1-\delta) f(\bar{a}_2) - \eta$, for arbitrary $\eta > 0$. This holds because $\bar{a}_2$ is bounded above in any SE. In any case, the following displayed inequalities are correct.

in $\Gamma_{PUB}|_M$: given $a_1, a_{2,1}, a_{2,3}, a_3 \in \mathbb{R}_+$, let

$$\sigma_{1,2}\left(h_1^t\right) = \begin{cases} a_1 \text{ if } \left(a_{1,2,\tau} = a_1, a_{2,1,\tau} = a_{2,1}, a_{2,3,\tau} = a_{2,3}, a_{3,2,\tau} = a_{3,2}\right) \text{ for all } \tau < t, \\ 0 \text{ otherwise} \end{cases}$$
$$\sigma_{2,1}\left(h_2^t\right) = \begin{cases} a_{2,1} \text{ if } \left(a_{1,2,\tau} = a_1, a_{2,1,\tau} = a_{2,1}, a_{2,3,\tau} = a_{2,3}, a_{3,2,\tau} = a_{3,2}\right) \text{ for all } \tau < t, \\ 0 \text{ otherwise} \end{cases}$$
$$\sigma_{2,3}\left(h_2^t\right) = \begin{cases} a_{2,3} \text{ if } \left(a_{1,2,\tau} = a_1, a_{2,1,\tau} = a_{2,1}, a_{2,3,\tau} = a_{2,3}, a_{3,2,\tau} = a_{3,2}\right) \text{ for all } \tau < t, \\ 0 \text{ otherwise} \end{cases}$$
$$\sigma_{3,2}\left(h_3^t\right) = \begin{cases} a_3 \text{ if } \left(a_{1,2,\tau} = a_1, a_{2,1,\tau} = a_{2,1}, a_{2,3,\tau} = a_{2,3}, a_{3,2,\tau} = a_{3,2}\right) \text{ for all } \tau < t, \\ 0 \text{ otherwise} \end{cases}.$$

It is straightforward to check that this strategy profile is a SE if and only if $a_1, a_{2,1}, a_{2,3}, a_3 \in A_{i,j}$ and

$$\begin{aligned} \frac{1}{2}\left(f\left(a_1, a_{2,1}\right) - (1-\delta) f\left(0, a_{2,1}\right)\right) - a_1 &\geq 0 \\ \frac{1}{2}\left(f\left(a_3, a_{2,3}\right) - (1-\delta) f\left(0, a_{2,3}\right)\right) - a_3 &\geq 0 \\ \frac{1}{2}\left(f\left(a_{2,1}, a_1\right) + f\left(a_{2,3}, a_3\right) - (1-\delta)\left(f\left(0, a_1\right) + f\left(0, a_3\right)\right)\right) - \left(a_{2,1} + a_{2,3}\right) &\geq 0. \end{aligned}$$

I now maximize $u_{1,2} = \frac{1}{2} f\left(a_1, a_{2,1}\right) - a_1$ over the inequality constraints, temporarily ignoring the constraint that $a_1, a_{2,1}, a_{2,3}, a_3 \in A_{i,j}$. It is clear that this is achieved by first maximizing $\frac{1}{2}\left(f\left(a_{2,3}, a_3\right) - (1-\delta) f\left(0, a_3\right)\right) - a_{2,3}$ over $\left(a_{2,3}, a_3\right)$ subject to $\frac{1}{2}\left(f\left(a_{2,3}, a_3\right) - (1-\delta) f\left(a_{2,3}, 0\right)\right) - a_3 \geq 0$, and then maximizing $\frac{1}{2} f\left(a_1, a_{2,1}\right) - a_1$ over $\left(a_{2,1}, a_1\right)$ subject to the remaining inequality constraints. Denote the solution to the first maximization problem by $\left(a_{2,3}^{**}, a_3^{**}\right)$, and let $\Delta \equiv \frac{1}{2}\left(f\left(a_{2,3}^{**}, a_3^{**}\right) - (1-\delta) f\left(0, a_3^{**}\right)\right) - a_{2,3}^{**}$ be the slack in player 2's incentive constraint coming from his relationship with player 3. Note that $\delta > \frac{2-f_i(0,0)}{f_i(0,0)}$ is a sufficient condition for $\Delta > 0$, as it implies the existence of a number $a_{i,j}$ such that $\frac{1}{2}\left(f\left(a_{i,j}, a_{i,j}\right) - (1-\delta) f\left(0, a_{i,j}\right)\right) - a_{i,j} > 0$. The reduced program is

$$\max_{a_1, a_{2,1}} \frac{1}{2} f\left(a_1, a_{2,1}\right) - a_1$$

$$s.t.\ \ \frac{1}{2}\left(f\left(a_1, a_{2,1}\right) - (1-\delta) f\left(a_1, 0\right)\right) - a_{2,1} + \Delta \geq 0.$$

Call this Program 2. It is now clear that the value of Program 2 is greater than the value of Program 1, and hence greater than $\bar{u}_1^{PRI}$; in particular, letting (a_1^*, a_2^*) denote the solution to Program 1, setting $a_1 = a_1^*$ and $a_{2,1} = a_2^* + \Delta$ satisfies the constraints of Program 2 and yields a strictly higher value in the objective. Finally, letting $\left(a_1^{**}, a_{2,1}^{**}, a_{2,3}^{**}, a_3^{**}\right)$ denote the solution to Program 2, it is clear that there is a set of vectors $(a_1, a_{2,1}, a_{2,3}, a_3)$ including $\left(a_1^{**}, a_{2,1}^{**}, a_{2,3}^{**}, a_3^{**}\right)$ with non-empty interior such that for any such vector $\frac{1}{2} f\left(a_1, a_{2,1}\right) - a_1 > \bar{u}_1^{PRI}$ and the resulting grim-trigger strategy profile satisfies the inequality constraints. Hence, if $A_{i,j}$ is ε-dense for small enough ε, then, noting that $a_1^{**}, a_{2,1}^{**}, a_{2,3}^{**}, a_3^{**} \leq \hat{a}_{i,j}$ (as can be checked), there exists a vector $(a_1, a_{2,1}, a_{2,3}, a_3)$ with $a_1, a_{2,1}, a_{2,3}, a_3 \in A_{i,j}$ such that the resulting grim-trigger strategy profile is a SE and $\frac{1}{2} f\left(a_1, a_{2,1}\right) - a_1 > \bar{u}_1^{PRI}$.

11.8 Proof of Proposition 2

I start by showing that there exists (Y, m^0) such that $E_{PRI}^{\$}\left(Y, m^0\right) \supseteq E_{PUB}$, that is, showing that Theorem 1 applies to the trading favors game. The proof is almost the same as proof of Theorem 1, but with different play in the communication phase.

Let $Y_{i,j} = \left(N^2 \times \{\emptyset, 0, 1\}\right) \cup \{0\}$. Let $m_i^0 = 8n' \left|N^2\right|$ for all $i \in \{1, \ldots, n'\}$ and $m_i^0 = 0$ for all $i \in \{n'+1, \ldots, n\}$. Play in the communication phase is as follows:

Reporting Subphase: There are up to $3(n-1)$ rounds of reporting in which players report whether the last favor was done and who currently has the opportunity to do a favor. Suppose the current opportunity is for i to do a favor for j. Let t_i^- be last time at which player i observed an opportunity to do a favor or sent or received a message, and suppose that player i observed or was told that the opportunity was for i_i' to do a favor for j_i'; if player i did not receive a report on this, player i is at an off-path history and play is described below. In round 1 player i sends message $\left(\beta_{i,j}, z_{\{i',j'\}_i, t_i^-} \cup \{\emptyset\}\right)$, where she sends $z_{\{i',j'\}_i, t_i^-} \in \{0,1\}$ if $\{i', j'\}_i \in \{\{i,j\}, (j,i)\}$ and sends $\emptyset$ otherwise; here, $\beta_{i,j}$ indicates that the current opportunity is for i to do a favor for j, $z = 1$ indicates that the previous favor was done, and $z = 0$ indicates that it was not done. In future rounds, each player i'' passes on $\hat{\beta}_{i,j}$, adds report

$z_{\{i',j'\}_{i''},t^{-}_{i''}}$ if $\{i',j'\}_{i''} \in \{(i'',j''),(j'',i'')\}$, and passes on $\hat{z}_{\{i',j'\}_{i''},t^{-}_{i''}}$ if he has received consistent reports about $z_{\{i',j'\}_{i''},t^{-}_{i''}}$. Each player sends a message according to this protocol $2(n-1)$ times (so there at most $3(n-1)$ rounds total). Note that on-path all players are informed of whether the last favor was done and of the identity of the current favor opportunity. Also, all players can infer the time and identity of the previous favor opportunity, as all players receive messages whenever there is a favor opportunity. In particular, all players are informed of everything they would observe in Γ_{PUB}.

Confirmation Subphase: Assign a number in $1,\ldots,|2N^2|$ to every element of $N^2 \times \{0,1\}$, corresponding to whether the previous favor was done and who currently has the opportunity to do a favor. Play in the confirmation subphase is as in the proof of Theorem 1, with transfers now corresponding to elements of $N^2 \times \{0,1\}$ rather than elements of Z.

Off-Path Play: As in Theorem 1, but replace "play $\alpha^*_{i,j'}$ in every subsequent period" with "never do another favor for j'."

The proof that this is a SE profile is similar to the proof of Theorem 1 and is omitted. In particular, note that the only "action histories" of a player j are ones where he has the opportunity to do a favor, and player j initiates a communication phase at these histories. So if player j does not receive a message in some communication phase, he learns about this before doing another favor. Hence, Lemma 1 continues to hold: for any deviator i and player j, j either plays toward i as he would had i conformed or he minmaxes i.

Now, given the result that $E^{\$}_{PRI} \supseteq E_{PUB}$, the proof that money is strongly essential if L contains a truly nice subnetwork follows is nearly identical to the proof of Theorem 2, and is omitted.

11.9 Proof of Theorem 4

I claim that $E_{PUB}|_M \backslash \operatorname{co}\left(E^{LCTE}_{PRI}|_M\right) \neq \emptyset$ for any subtree $M = \{(1,2),(2,3)\}$ of size three. The theorem then follows from Proposition 2.

I show that $E_{PUB}|_M \backslash \operatorname{co}\left(E_{PRI}^{LCTE}|_M\right) \neq \emptyset$ by building on the analysis of the two-player trading favors game due to Hauser and Hopenhayn (2010). They develop a differential characterization of the PPE set in this game. In particular, they show that the Pareto frontier in the two-player game, $W(v)$ (where v is player 1's continuation value) is given by

$$rW(v) = \max_{a_1, a_2, v^1, v^2, \dot{v}} a_1 b - a_2 c + W\left(v^1\right) - W(v) + W\left(v^2\right) - W(v) + W'(v)\,\dot{v} \tag{4}$$

subject to

$$\begin{aligned} rv &= a_2 b - a_1 c + v^1 - v + v^2 - v + \dot{v} \\ v &= v^1 - a_1 c \\ W(v) &= W\left(v^2\right) - a_2 c. \end{aligned}$$

In this formulation, v^1 is player 1's continuation value when she does a favor for player 2 (or reveals an opportunity to do a favor but the public randomization is such that she does not actually do it), v^2 is player 1's continuation value when player 2 does a favor for her, $\dot{v}$ is the drift in player 1's continuation value, and a_i is the probability that player i does a favor when she gets an opportunity to do so. The first constraint is the promise-keeping constraint, and the second and third constraints are player 1's and 2's incentive constraints, which Hauser and Hopenhayn show hold with equality at a solution. Hauser and Hopenhayn also show that W is weakly concave, that $W'(v) \in \left[-\frac{b}{c}, -\frac{c}{b}\right]$ almost everywhere, and that at a solution a_1 and a_2 are given by

$$\begin{aligned} a_1 &= \min\left\{\frac{v^h - v}{c}, 1\right\} \\ a_2 &= \min\left\{\frac{W(0) - W(v)}{c}, 1\right\}, \end{aligned}$$

where $v^h = W^{-1}(0)$.[41]

I now show that $E_{PUB}|_M \backslash \operatorname{co}\left(E_{PRI}^{LCTE}|_M\right) \neq \emptyset$. I use the following technical lemma about the two-player game:

[41] To be precise, there is a single solution where the incentive constraints bind and x_1 and x_2 are given by these equations.

Lemma 3 *$W'(v) < -\frac{c}{b}$ for some $v \in (0, b)$.*

This lemma would be immediate if W were strictly concave, but Hauser and Hopenhayn only establish weak concavity (while conjecturing strict concavity).

Proof of Lemma 3. Suppose, toward a contradiction, that $W'(v) = -\frac{c}{b}$ for all $v \in (0, b)$. By symmetry of the two-player game, $W'\left(v^h\right) = -\frac{b}{c}$. In particular, by weak concavity of W there exists $v^* \in \left(b, v^h\right)$ such that $W'(v) = -\frac{c}{b}$ for all $v < v^*$ and $W'(v) < -\frac{c}{b}$ for almost all $v > v^*$. Now let v satisfy $v \in (0, v^*)$ and $v + c \in \left(v^*, v^h\right)$. Then, by (4) and the fact that $W'(v) = -\frac{c}{b}$,

$$\begin{aligned} rW(v) &= a_1 b - a_2 c + W\left(v^1\right) - W(v) + W\left(v^2\right) - W(v) \\ &\quad -\frac{c}{b}\left(rv - a_2 b + a_1 c - v^1 + v - v^2 + v\right). \end{aligned}$$

Now since $W(v') = -\frac{c}{b}$ for all $v' < v$, and $v^2 < v$, it follows that $W(v^2) - W(v) = -\frac{c}{b}(v^2 - v)$. In addition, $a_1 c = v^1 - v$. So the above equation becomes

$$W(v) = \frac{a_1 b + W(v^1) - W(v)}{r} - \frac{c}{b}v. \tag{5}$$

Now since $v + c < v^h$, it follows from Hauser and Hopenhayn that $a_1 = 1$ for all v' in a neighborhood of v (for some solution), and hence that $v'^1 = v' + c > v^*$ for all such v'. Since $W'(v') = -\frac{c}{b}$ for all $v' < v^*$ and $W'(v') < -\frac{c}{b}$ for all $v' > v^*$, it follows that $W(v'^1) - W(v') = W(v' + c) - W(v')$ is strictly decreasing with non-vanishing derivative in a neighborhood of v. Therefore, it follows from (5) that W cannot be differentiable at v with $W'(v) = -\frac{c}{b}$, completing the proof of the lemma. ■

Turning to the three player game, note that $W(v_1) + W(v_3)$ is the Pareto frontier in $\Gamma_{PRI}|_M$ (restricting to LCTE), and let $W(v_1, v_3)$ be the Pareto frontier in $\Gamma_{PUB}|_M$. I show that there exist v_1, v_3 such that $W(v_1, v_3) > W(v_1) + W(v_3)$, which completes the proof. To prove this, I first introduce some notation. Let a_1 be the probability that 1 does a favor for 2 and let a_{21} be the probability that 2 does a favor for 1, and similarly for a_3 and a_{23}. Let v_1^1 be player 1's continuation value when 1 does a favor (for 2), let v_1^{21} be player 1's continuation value when 2 does a favor for 1, and so on for $v_1^{23}, v_1^3, v_3^1, v_3^{21}, v_3^{23}, v_3^3$. Now let $\tilde{W}(v_1, v_3) =$

$W(v_1)+W(v_3)$, $\tilde{a}_1=\min\left\{\frac{v^h-v_1}{c},1\right\}$, $\tilde{a}_{21}=\min\left\{\frac{2W(0)-W(v_1)-W(v_2)}{c},1\right\}$, $\tilde{v}_1^1=v_1+\tilde{a}_1c$, $\tilde{v}_1^3=v_1$, $\tilde{v}_1^{21}=\max\left\{W^{-1}\left(W(v_1)+\tilde{a}_{21}c\right),0\right\}$, $\tilde{v}_1^{23}=W^{-1}\left(W(v_1)+\tilde{a}_{23}c-\left(W\left(\tilde{v}_3^{23}\right)-W(v_3)\right)\right)$, and symmetrically for $\tilde{a}_3$, $\tilde{a}_{23}$, $\tilde{v}_3^3$, $\tilde{v}_3^1$, $\tilde{v}_3^{23}$, and $\tilde{v}_3^{21}$, and finally let

$$\begin{aligned}d\tilde{W}(v_1,v_3) &= W'(v_1)\left(rv_1-\tilde{a}_{21}b+\tilde{a}_1c-\tilde{v}_1^1-\tilde{v}_1^{21}-\tilde{v}_1^{23}-\tilde{v}_1^3+4v_1\right)\\ &\quad+W'(v_3)\left(rv_3-\tilde{a}_{23}b+\tilde{a}_3c-\tilde{v}_3^1-\tilde{v}_3^{21}-\tilde{v}_3^{23}-\tilde{v}_3^3+4v_3\right).\end{aligned}$$

I now claim that there exist v_1,v_3 such that

$$\begin{aligned}r\tilde{W}(v_1,v_3) &< (\tilde{a}_1+\tilde{a}_3)b-(\tilde{a}_{21}-\tilde{a}_{23})c+\tilde{W}\left(\tilde{v}_1^1,\tilde{v}_3^1\right)+\tilde{W}\left(\tilde{v}_1^{21},\tilde{v}_3^{21}\right)\\ &\quad+\tilde{W}\left(\tilde{v}_1^{23},\tilde{v}_3^{23}\right)+\tilde{W}\left(\tilde{v}_1^3,\tilde{v}_3^3\right)-4\tilde{W}(v_1,v_3)+d\tilde{W}(v_1,v_3). \qquad (6)\end{aligned}$$

To see this, recall that

$$\begin{aligned}r\tilde{W}(v_1,v_3) &= r\left(W(v_1)+W(v_3)\right)=(a_1+a_3)b-(a_{21}-a_{23})c+W\left(v_1^1\right)+W\left(v_1^{21}\right)-2W(v_1)\\ &\quad+W\left(v_3^{23}\right)+W\left(v_3^3\right)-2W(v_3)+W'(v_1)\dot{v}_1+W'(v_3)\dot{v}_3, \qquad (7)\end{aligned}$$

by (4). So the claim follows if the right-hand side of (6) is greater than the right-hand side of (7). Working through the definitions (including observing that $\tilde{a}_1=a_1$, $\tilde{v}_1^1=v_1^1$, $\tilde{v}_1^3=v_1^3$, and $\tilde{v}_1^{21}=v_1^{21}$, and similarly for player 3), this is the case if

$$\left|W'(v_1)\left(\tilde{a}_{21}b+\tilde{v}_1^{23}\right)+W'(v_3)\left(\tilde{a}_{23}b+\tilde{v}_3^{21}\right)\right|>\left|W'(v_1)(a_2b+v_1)+W'(v_3)(a_{23}b+v_3)\right|. \qquad (8)$$

Now $\tilde{a}_{21}\geq a_2$, with strict inequality if $v_1<b$ and $v_3>0$, and similarly for $\tilde{a}_{23}$. Let $\tilde{a}_{23}=a_{23}$, and differentiate the left-hand side of (8) with respect to $\tilde{a}_{21}$ at $\tilde{a}_{21}=a_2$. The derivative equals

$$W'(v_1)b+W'(v_3)\left(\frac{c}{W'(v_3)}\right)=W'(v_1)b+c.$$

So taking any $v_3>0$ and any $v_1\in(0,b)$ such that $W'(v_1)<-\frac{c}{b}$—which exists, by Lemma 3—implies that this derivative is negative and that $\tilde{a}_{21}>a_2$, and hence that (8)—and therefore (6)—is satisfied. Therefore, Hauser and Hopenhayn's optimality equation (4) (or, more precisely, its immediate generalization to the three-player game $\Gamma_{PUB}|_M$) is not satisfied by $\tilde{W}$, while the constraints are satisfied are satisfied by $\tilde{W}$. So there exist v_1, v_3 such that $W(v_1, v_3) > W(v_1) + W(v_3)$, completing the proof.

References

[1] Abreu, D., D. Pearce, and E. Stacchetti (1990), "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica*, 58, 1041-1063.

[2] Ahn, I. and M. Suominen (2001), "Word-Of-Mouth Communication and Community Enforcement," *International Economic Review*, 42, 399-415.

[3] Ali, S.N. and D. Miller (2012), "Ostracism," mimeo.

[4] Aliprantis, C.D., G. Camera, and D. Puzzello (2007), "Anonymous Markets and Monetary Trading," *Journal of Monetary Economics*, 54, 1905-1928.

[5] Araujo, L. (2004), "Social Norms and Money," *Journal of Monetary Economics*, 51, 241-256.

[6] Aumann, R.J. and S. Hart (2003), "Long Cheap Talk," *Econometrica*, 71, 1619-1660.

[7] Balmaceda F. and J. Escobar (2011), "Self Governance in Social Networks of Information Transmission," mimeo.

[8] Ben-Porath, E. and Kahneman, M. (1996), "Communication in Repeated Games with Private Monitoring," *Journal of Economic Theory*, 70, 281-297.

[9] Compte, O. (1998), "Communication in Repeated Games with Imperfect Private Monitoring," *Econometrica*, 66, 597-626.

[10] Escobar, J. and J. Toikka (2012), "Efficiency in Games with Markovian Private Information," mimeo.

[11] Ellison, G. (1994), "Cooperation in the Prisoner's Dilemma with Anonymous Random Matching," *Review of Economic Studies*, 61, 567-588.

[12] Fudenberg, D. and D.K. Levine (1994), "Efficiency and Observability with Long-Run and Short-Run Players," *Journal of Economic Theory*, 62, 103-135.

[13] Fujiwara-Greve, T., M. Okuno-Fujiwara, and N. Suzuki (2012), "Voluntarily Separable Prisoner's Dilemma with Reference Letters," *Games and Economic Behavior*, 74, 504-516.

[14] Hahn, F. (1973), "On the Foundations of Monetary Theory," in *Essays in Modern Economics*, edited by M. Parkin with A. R. Nobay, New York: Barnes and Noble.

[15] Hauser, C. and H. Hopenhayn (2010), "Trading Favors: Optimal Exchange and Forgiveness," mimeo.

[16] Hörner, J. and W. Olszewski (2006), "The Folk Theorem for Games with Private Almost-Perfect Monitoring," *Econometrica*, 74, 1499-1544.

[17] Kandori, M. (1992), "Social Norms and Community Enforcement," *Review of Economic Studies*, 59, 63-80.

[18] Kandori, M. and H. Matsushima (1998), "Private Observation, Communication and Collusion," *Econometrica*, 66, 627-652.

[19] Kiyotaki, N. and R. Wright (1993), "A Search-Theoretic Approach to Monetary Economics," *American Economic Review*, 83, 63-77.

[20] Kocherlakota, N.R. (1996), "Implications of Efficient Risk Sharing without Commitment," *Review of Economic Studies*, 63, 595-609.

[21] Kocherlakota, N.R. (1998), "Money is Memory," *Journal of Economic Theory*, 81, 232-251.

[22] Kocherlakota, N. and N. Wallace (1998), "Incomplete Record-Keeping and Optimal Payment Arrangements," *Journal of Economic Theory*, 81, 272-289.

[23] Lagos, R. and R. Wright (2008), "When is Money Essential? A Comment on Aliprantis, Camera, and Puzzello," mimeo.

[24] Lippert, S. and G. Spagnolo (2011), "Networks of Relations and Word-of-Mouth Communication," *Games and Economic Behavior*, 72, 202-217.

[25] Milgrom, P.R., D.C. North, and B.R. Weingast (1990), "The Role of Institutions in the Revival of Trade: The Law Merchant, Private Judges, and the Champagne Fairs," *Economics and Politics*, 2, 1-23.

[26] Mishkin, F.S. (1998), *The Economic of Money, Banking, and Financial Markets*, 5th ed., Reading: Addison-Wesley.

[27] Möbius, M.M. (2001), "Trading Favors," mimeo.

[28] Okuno-Fujiwara, M. and A. Postlewaite (1995), "Social Norms and Random Matching Games," *Games and Economic Behavior*, 9, 79-109.

[29] Renault, J. and T. Tomala (1998), "Repeated Proximity Games," *International Journal of Game Theory*, 27, 539-559.

[30] Wallace, N. (2001), "Whither Monetary Economics?" *International Economic Review*, 42, 847-869.